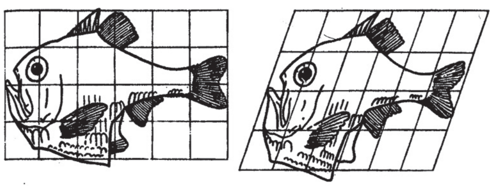
Changing Shapes With Matrices©

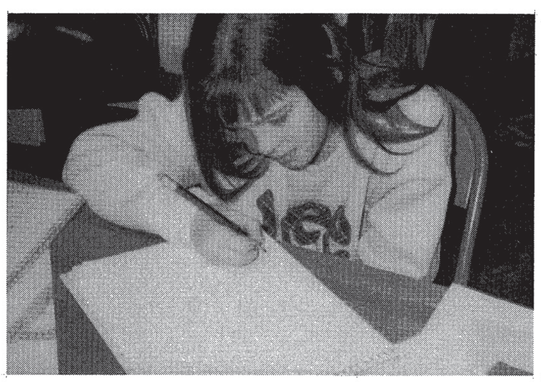
by Don Cohen

- a book for ages 7 to adult - for people into computer graphics, morphs, how living things differ in shape, or just in the fun and power of being able to change a 2-D shape, oneself, quickly and easily.

From the author of the successful books "Calculus By and For Young People (ages 7, yes 7 and up)", and "Calculus By and For Young People - Worksheets", the effective videotapes "Infinite Series By and For 6 year olds an up" and "Iteration to Infinite Sequences with 6 to 11 year olds", the poster "A Map to Calculus", Math By Mail and Co- founder and Teacher of <u>The Math Program</u>



Valorie made up a matrix which caused a change (or transformation) in the shape of a dog, similar to one D'Arcy Thompson talked about with fish in his 1917 book <u>On Growth and Form</u> , also shown in the book <u>The Art of Graphics for the IBM PC</u> written in 1986! Exciting stuff!



A copy of Don's watercolor painting of a chambered Nautilus

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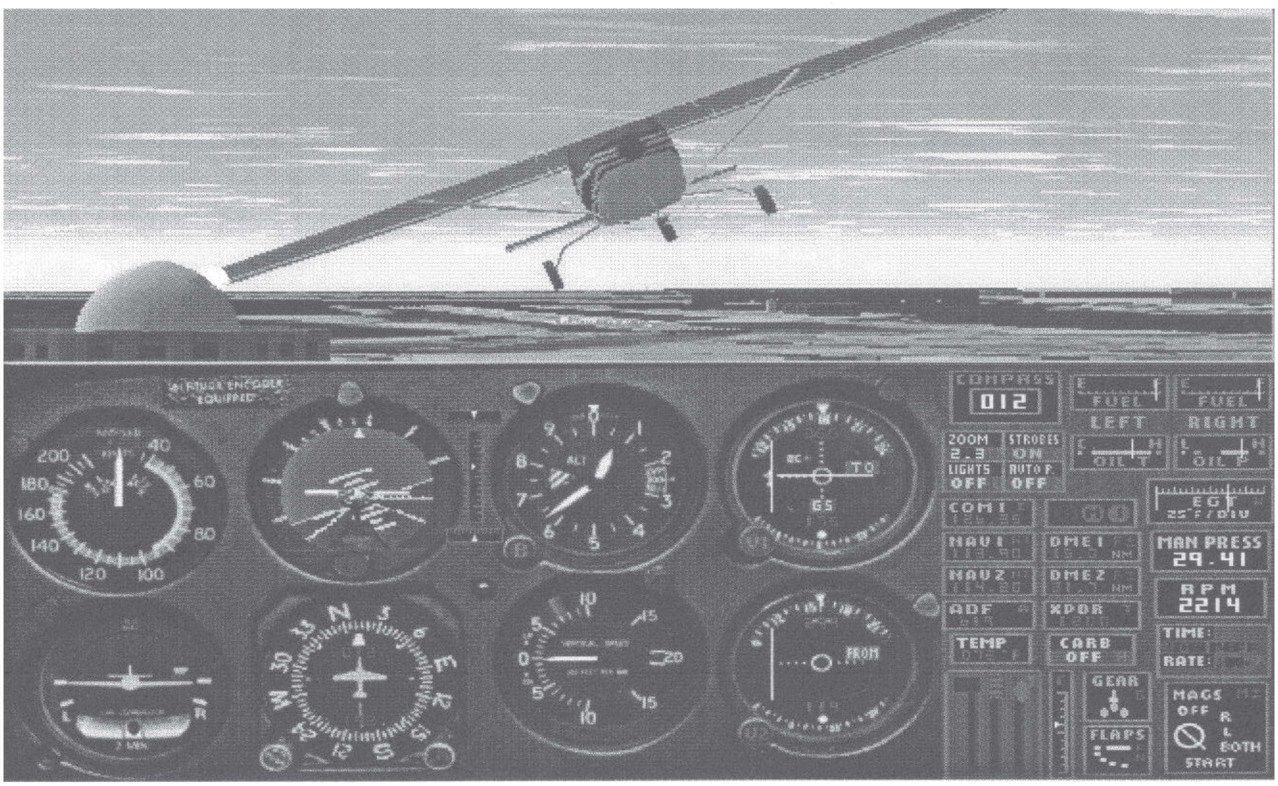

In his "Flight Simulator"® computer program, Bruce Artwick changes the heading, pitch and bank of an airplane, rotations about three different axes, as well translations and scaling to move the plane from 3-D to a 2-D screen. He uses **matrices** as a base, to do these transformations and for us to simulate the plane's flight.

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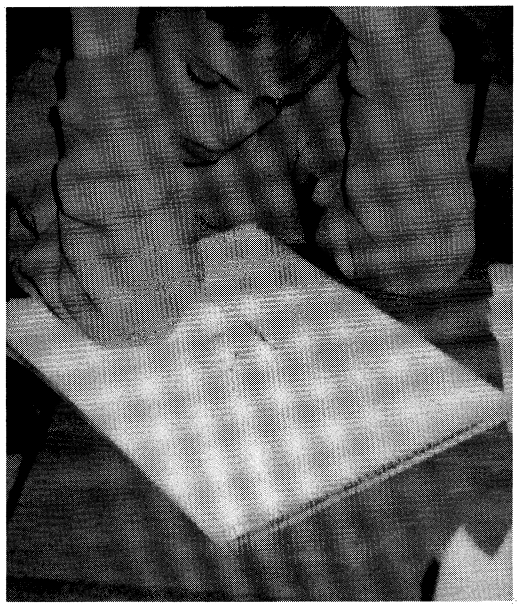
From the author of the successful books "Calculus By and For Young People (ages 7, yes 7 and up)", and "Calculus By and For Young People - Worksheets", the effective videotapes "Infinite Series By and For 6 year olds an up" and "Iteration to Infinite Sequences with 6 to 11 year olds", the poster "A Map to Calculus", Math By Mail and Co- founder and Teacher of The Math Program



Mark, age 7, works on changing the dog with his matrix

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Acknowledgments

I would like to thank Dr. Robert B. Davis, at Rutgers now, for showing me how the candy-store arithmetic can make multiplying matrices so easy for all of us to understand. A short story-- A math coordinator from Rochester NY and I were observing Bob for a day (in Scarsdale, NY, circa 1960), working with children and teachers using the materials he developed as part of The Madison Project (see bibliography) . In the class we observed he introduced matrices to a 5th grade class. After this class Bob found out he had to leave and asked us if we would teach a couple of classes and work with a group of teachers after school. Now the thing was, both of us had matrices in college, but neither one of us understood them. Watching Bob do the candy-store arithmetic, enabled us to do this with reduced apprehension.

I want to thank Bruce Artwick who helped me understand how he used matrix transformations (3-D and 2-D) to move the airplane in his best-selling "Flight Simulator" computer program. The picture of the airplane on the inside front cover was provided by Bruce Artwick Organization, Ltd.

I am continuously inspired by D'Arcy Thompson's book <u>On Growth and Form</u> in which he shows how transformations can applied in nature, although he didn't use matrices. The fish pictures on the cover and in chapter 5 are reprinted with permission of Cambridge University Press.

I want to thank Theo Gray, who as an 11 year old student of mine, did such fine work with matrices (see ch. 4) that he got me doing more with other young people. Now, about 18 years later, as a Director of User Interfaces for Wolfram Research Inc., Theo wrote a procedure in *Mathematica* ® to show the transformations of the dog that resulted from using the 81- 2x2 matrices with contain only 1's, 0's or $^{-}1$'s, and then printed them for this book. He also printed the non-linear dog in ch. 5, **wow**!

I want to thank the parents who have sent their children to us at The Math Program. This is a unique program for teaching mathematics which Jerry Glynn and I founded and have taught in, for over 19 years! The Math Program has enabled us to work on mathematics with youngsters of ages 3 to 73, of all abilities, in small groups, in our own home, and enjoy it. Because of this way of teaching, we have each been able to contribute to the teaching of mathematics, nationally, as well as internationally.

Most importantly, I want to thank Marilyn, my wife of 42 years, for encouraging me and putting up with yet another of my creations.

Preface

I have been tremendously encouraged by the use of my books "Calculus By and For Young People (ages 7, yes 7 and up)" and "Calculus By and For Young People-Worksheets", my two videotapes, and my poster "A Map to Calculus" -- by parents, children and teachers throughout the world. I hear everyday from parents who are looking for math materials that go beyond rote learning and boredom. I am proud to have been able to expand the possibilities for young people, not only to understand some real mathematics, but to create it as well, and more important, to enjoy it.

Why transformations and why matrices?

I decided to write this little book to share my interest and excitement about the ideas of **changing shapes** (geometric transformations) **using matrices**-- with parents, students, teachers, and teachers of teachers. The reasons I think this is an important piece of curriculum is that

1) In the process of doing this work, my students do <u>a lot of arithmetic</u> and are always encouraged to look for patterns.
2) It involves coordinate or Cartesian geometry, which brings together arithmetic, algebra and geometry -a very important and useful mathematical idea from Descartes, circa 1630.
3) A matrix (by Cayley, circa 1850) is an array of numbers which can represent a rule. This allows us to take a point and change its position just by multiplying and adding.
4) Matrices are used in computer programming and computer graphics, <u>today</u>. For example, changing the heading, pitch, and bank of an airplane are all rotations about three different axes and matrices are used to make these changes in Bruce Artwick's "Flight Simulator" computer program.
5) Transformations are changes which can be used to describe how animals, shells, and plants relate to each other
6) One can see a **picture** of how each matrix changes the original shape- it's a very visual part of mathematics! Some of my high school students can manipulate matrices using their calculators, but they rarely understand what they are doing and don't get to appreciate the power of the matrices.

When I saw drawings from D'Arcy Thompson's book 'On Growth and Form', first published in 1917, used in a book on computer graphics published in 1986, I knew I was doing something important and headed in the right direction!

Some other words that go with the transformation of shapes: change, mutation, metamorphosis, morphins (power rangers), modify, alteration, distortion, modulation (shapes of waves), growth (proportion, linear and non-linear), gravitational change, shear (deformation, compression, forces), rotation, reflection (symmetry), translation, scaling (enlargement, magnification), and tessellation. These have very important ramifications, not only in mathematics, but in other aspects of life as well.

There are also **algebraic transformations**, used in solving equations, that are dealt with in my book "Calculus By and For Young People-Worksheets". Besides the **rectangular** coordinates which will be used here, polar or radial coordinates can also be used (as in leaf changes- see D'Arcy Thompson's book).

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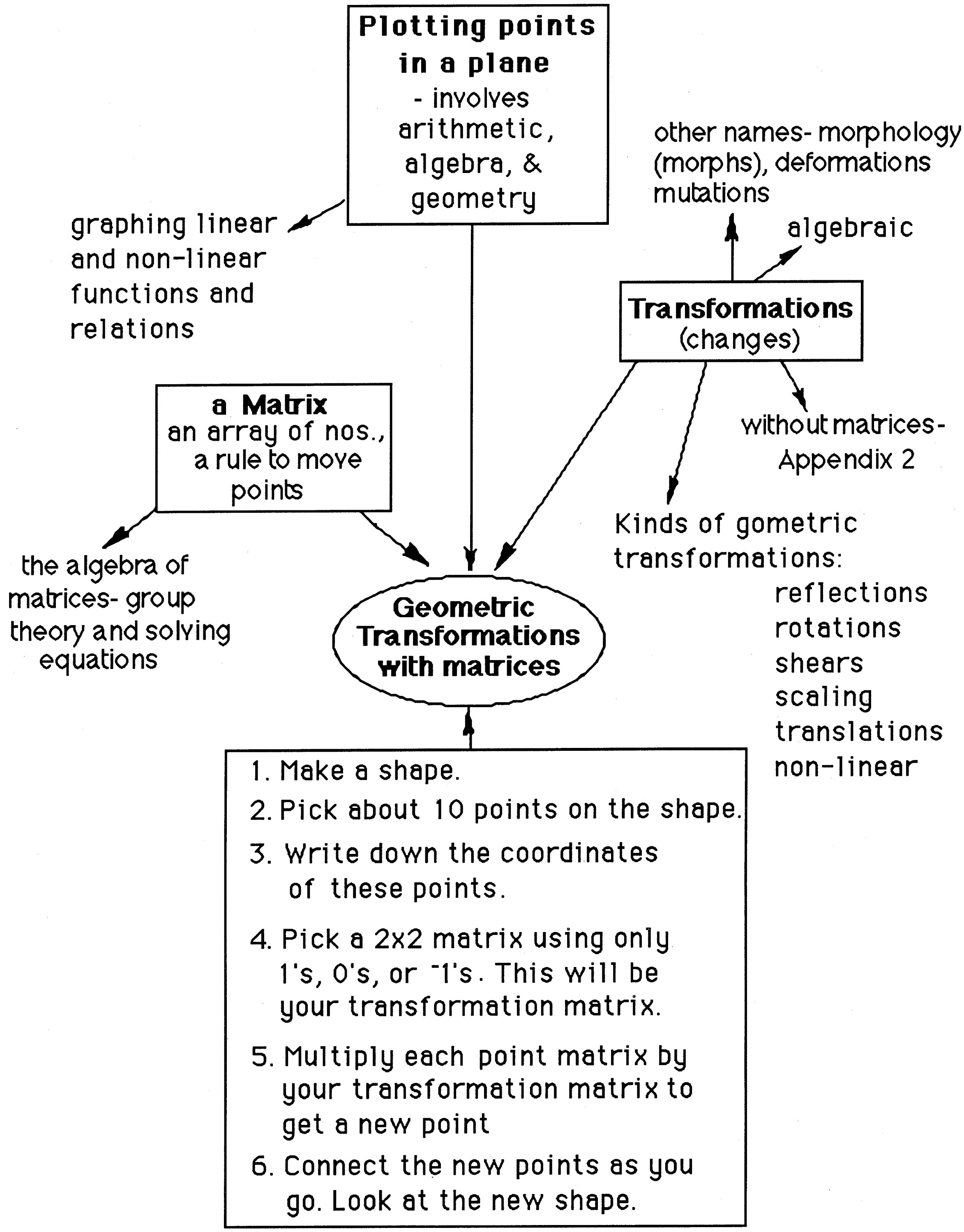
A Map to Transfomations

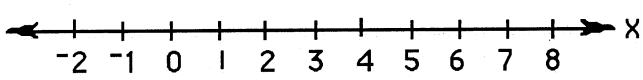
also connects to "A Map to Calculus"



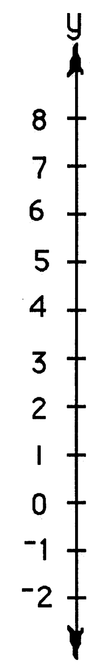
Ch. 1: Plotting points

We're going to graph the equation x + y = 7. The problem here is to find two numbers that add up to 7 or find some pairs of numbers that make x + y =7 true. If we put 1 in for x and 6 in for y we get 1+6 =7, which is true. We put 1 and 6 in the table below on the right. There are 2 number lines (the thicker lines), a horizontal one for the x-number, the first number in the pair, and a vertical number line for the y-number, the second number in the pair.

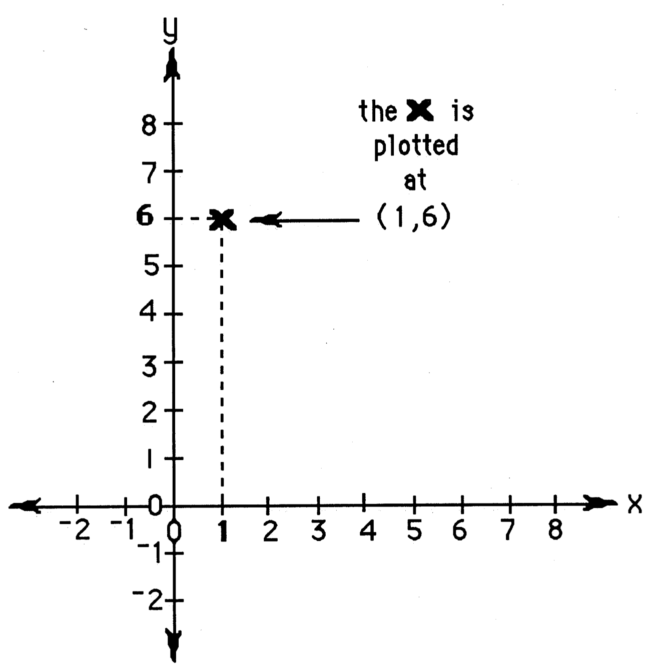
x	y
1	6
5	2
7	0



Think of the graph paper as two number lines, a horizontal one, called the x-axis and a vertical one called the y-axis. Then put them together at (0,0).

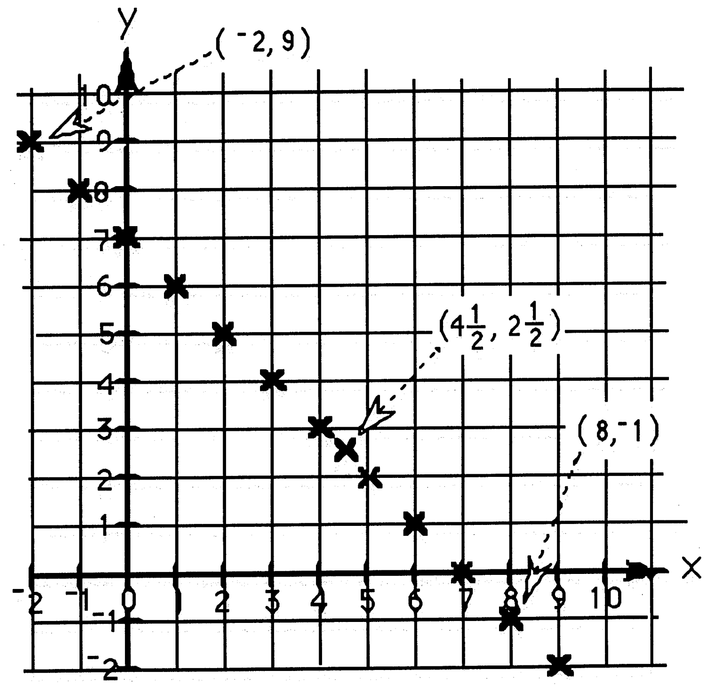


This gives our graphing board, called the Cartesian plane, after Descartes (pronounced day-**cart**,). Each pair of numbers corresponds to a point on the graph. To plot the point (1,6), you go over 1 and up 6.



The points corresponding to the pairs of numbers in the table, the pairs that make x + y =7 true, are shown plotted on the graph paper below. Notice that the numbers for the x- and y- axes are written on the lines, not in the spaces as on a map.

$x + y = 7$



x	y
1	6
5	2
4	3
0	7
2	5
7	0
8	$^{-}1$
9	$^{-}2$
$^{-}2$	9
$4\frac{1}{2}$	$2\frac{1}{2}$

Most youngsters want to stop the "yellow brick road" or the pattern of the points at the x- and y- axes, but I push forward to get the negatives. If we put 8 in for x we have to go down 1 to get a point on the "yellow brick road", and we say negative one, written as $^{-}1$ for y. And 8+ $^{-}1$=7 is true. Notice that I say 'negative 1' and write it as $^{-}1$, **not** minus 1 and **not** written as -1 as is done on weather reports and in most textbooks. I distinguish carefully between the negative number and the operation of subtraction. Many youngsters have trouble in algebra classes because this is not clear. The point $4\frac{1}{2}$, $2\frac{1}{2}$ also makes $x + y = 7$ true, so I get into fractions early with young people.

Copy and use the $\frac{1}{2}$" graph paper in appendix. What happens if you graph **1.1** $x + y = 8$? How will it be different from the one above? How about **1.2** $x - y = 2$? **1.3** $x * y = 12$? **1.4** $\frac{x}{y} = 2$?

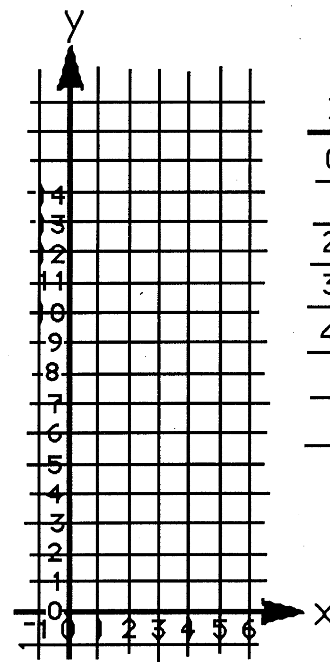
Use some $\frac{1}{4}$" graph paper for the following graphs.

Now graph: 2*x + 3 = y. Use the table of pairs of numbers that make this true and plot the points that go with them. Again look for patterns. There are lots of patterns in the numbers, find patterns on the graph.

Using a different color for each equation, make a table of numbers and graph these on the **same** graph paper as 2*x + 3 = y Write the equation, its table of numbers and make the graph, all in the same color, and each equation a different color: 2*x + 7 = y and 2*x + 1 = y . What's happening? What's changing and what's staying the same? Where does the adding number show up on the graph?

(There is more detail on this in my book "Calculus By and For Young People - Worksheets ").

2•x + 3 = y



x	y
0	3
1	5
2	7
3	9
4	11

Ch. 2: "Grocery store" arithmetic for multiplying matrices

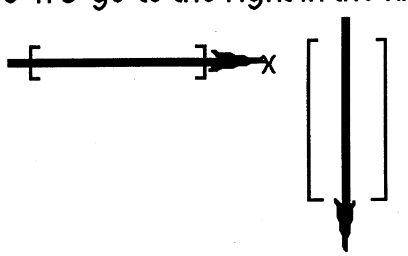
Pretend you go to the store to buy three things. What did you get? How many of each did you get? Write down the prices for each item. How much did it cost altogether?

Justin, 9 and Whitney 16 years old, worked together on this. Justin bought mustard (M), hot dogs (HD), and buns (B). He bought 2 bottles of mustard, 8 hot dogs, and 10 buns. He paid 80 cents for each bottle of mustard, 30 cents for each hot dog, and 12 cents for each bun. How much did this cost? And tell me how you get it. We'll keep track of this information in matrix form, the first matrix (a row matrix) will be how many items, the second one is the price matrix (a column matrix):

$$\begin{array}{c} \begin{array}{ccc} M & HD & B \end{array} \\ \begin{bmatrix} 2 & 8 & 10 \end{bmatrix} \end{array} \times \begin{bmatrix} 80 \\ 30 \\ 12 \end{bmatrix} = \underset{\text{Answer}}{\begin{bmatrix} 420 \end{bmatrix}}$$

$$2 \times 80 + 8 \times 30 + 10 \times 12 = 420$$

So we go to the right in the first matrix, and down



in the second matrix,
multiplying and **adding**
as we go.

The total price then is 420 cents and is a 1 row by 1 column matrix. If we keep the numbers simple and not worry about dollars, we can use whole numbers. The idea here is to learn about multiplying matrices, not decimals.

If Justin buys what he did on Monday, then Tuesday he buys 5 bottles of mustard, 7 hot dogs and 9 buns, what will his cost be for Tuesday, and how will he get it? The 420 remains the same for Monday. He will pay $5 \times 80 + 7 \times 30 + 9 \times 12 = 718$ cents, which goes in the second row of the answer matrix. Again we go to the right in the first matrix and down in the second matrix, multiplying and adding. This time it's going right in the <u>second</u> row of the first matrix and the answer is in the <u>second</u> row of the answer matrix.

$$\begin{array}{c} \text{Mon.} \\ \text{Tues.} \end{array} \begin{bmatrix} 2 & 8 & 10 \\ 5 & 7 & 9 \end{bmatrix} \times \begin{bmatrix} 80 \\ 30 \\ 12 \end{bmatrix} = \begin{bmatrix} 420 \\ \underset{?}{\_\_\_} \end{bmatrix}$$

2.1 Try this one:

what goes here ?

$$\begin{bmatrix} 6 & 2 & 9 \\ 3 & 7 & 5 \end{bmatrix} \times \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \end{bmatrix}$$

what goes here ?

How about this one:

$$\begin{bmatrix} 7 & 3 & 10 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ 3 & 8 \end{bmatrix} =$$

How many rows and columns are in this answer matrix ? And what numbers are in the matrix ?

We go to the right in the first matrix, then down in the second matrix, as usual.
The number in the **first** row of the left matrix times the numbers in the **first** column of the second matrix, added, will give the number in the **first** row and **first** column of the answer matrix.

$$\begin{bmatrix} 7 & 3 & 10 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 43 & \quad \end{bmatrix}$$

$$7 \times 1 + 3 \times 2 + 10 \times 3 = 43$$

The numbers in the **first** row of the left matrix times the numbers in the **second** column of the right matrix, added, will give the number in the **first** row and **second** column of the answer matrix.

$$\begin{bmatrix} 7 & 3 & 10 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 43 & 135 \end{bmatrix}$$

$$7 \times 4 + 3 \times 9 + 10 \times 8 = 135$$

2.2 Try this one:

$$\begin{bmatrix} 6 & 8 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 5 & 9 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} \_ & \_ \end{bmatrix}$$

What goes here ?

What goes here ?

2.3 Try this one:

$$\begin{bmatrix} 6 & 8 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 5 & 9 \end{bmatrix} =$$

Let's multiply a 2x2 matrix by a 2x2 matrix; there will be 4 numbers in the answer:

$$\begin{bmatrix} 2 & 4 \\ 9 & 8 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix}$$

Try it yourself before looking below.

$$\mathbf{2}\times 5 + \mathbf{4}\times 7 = 38$$

$$\mathbf{2}\times 3 + \mathbf{4}\times 6 = 30$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{9} & \mathbf{8} \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 30 & 38 \\ 75 & 101 \end{bmatrix}$$

$$\mathbf{9}\times 3 + \mathbf{8}\times 6 = 75$$

$$\mathbf{9}\times 5 + \mathbf{8}\times 7 = 101$$

2.4 How about this one- a 2X2 by a 2X2:

$$\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & 10 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix}$$

2.5 Can you generalize the multiplication of a 2x2 matrix by a 2x2 matrix?

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$$

2.6 Does the order of the matrices in the multiplication matter in other words, is multiplication of matrices commutative? Try reversing the two matrices above and see what happens.

We could ask other questions like: is the multiplication of matrices associative? Is there a matrix that acts like 1? 0? Getting to the geometry and the transformations has higher priority for us here and most of these questions will be answered as we go anyway.

2.7 This is the form of the multiplication of the matrices we will use for the transformations, try it:

$$\begin{bmatrix} 3 & 7 \end{bmatrix} \times \begin{bmatrix} 2 & 6 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} \quad\quad \end{bmatrix}$$

Generalizing the multiplication of a 1x2 by a 2x2 matrix:

$$\begin{bmatrix} X & Y \end{bmatrix} \times \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} AX + CY & BX + DY \end{bmatrix}$$

[X Y] is the matrix that represents the coordinates of a point (X,Y) on the original shape which we'll call OldX and OldY, then AX + CY is the NewX and BX + DY is the NewY, the coordinates of the point of the new, transformed shape. The answer above can be written like the rules given with the list of 81,

NewX = A * OldX + C * OldY
NewY = B * OldX + D * OldY

The 2x2 matrix will be your transformation matrix.

Ch. 3: Steps to do a transformation and a point-by-point restatement of Valorie's work

Steps to do a geometric transformations using matrices:

1. Make a shape. The shape I chose was the "dog" (simple, not too many points).
2. Pick out about 10 points on the shape.
3. Write down the coordinates of these points.
4. Number each of these points to make it easier to keep track of things.
5. Pick a 2x2 matrix using only 1's or 0's or $^-1$'s (for now). This will be your transformation

matrix. For example, Valorie chose the matrix: $\begin{bmatrix} 1 & ^-1 \\ 0 & 1 \end{bmatrix}$

6. Multiply each point (as a matrix) by your transformation matrix to get a new point.
7. Plot each new point and number it, keeping a correspondence with the old points.
8. Connect the new points as you go, using a color different from the original shape.
9. Complete all the new points and close the shape (if it's a closed figure), then look at the new shape carefully. Try to figure out what your matrix did to the original shape. Then ask questions about the situation.

Valorie' transformation matrix

① $\begin{bmatrix} 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

② $\begin{bmatrix} 3 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \end{bmatrix}$

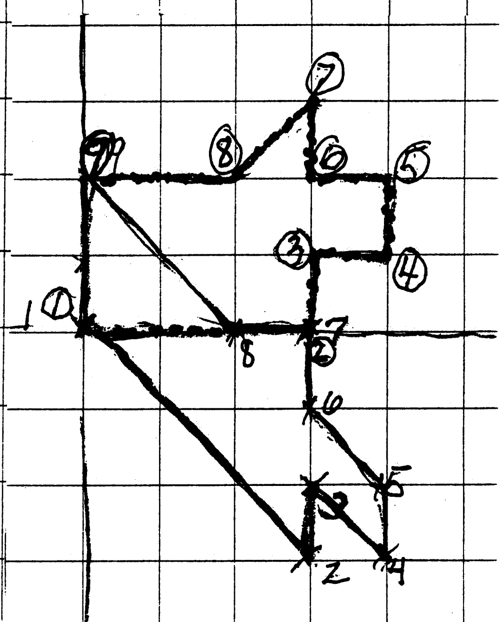
③ $\begin{bmatrix} 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \end{bmatrix}$

Valorie

⑦ $\begin{bmatrix} 3 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \end{bmatrix}$

⑧ $\begin{bmatrix} 2 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}$

⑨ $\begin{bmatrix} 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}$



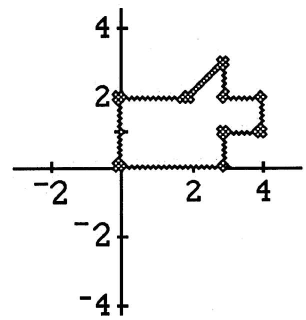
④ $\begin{bmatrix} 4 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \end{bmatrix}$

Looks like a shadow.

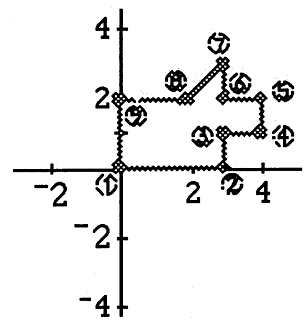
⑤ $\begin{bmatrix} 4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \end{bmatrix}$

⑥ $\begin{bmatrix} 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \end{bmatrix}$

A point-by-point restatement of Valorie's work:



The gray figure on the right is the "dog". The main points Valorie transformed have coordinates: {(0,0), (3,0), (3,1), (4,1), (4,2), (3,2), (3,3), (2,2), (0,2), (0,0)}



The light-circled numbers (for the old points) give the order in which Valorie transformed each point with her transformation matrix to get the new points. The new points were plotted and connected (heavy lines) to see what the new doggie looked like. Valorie plotted the new points in a different color.

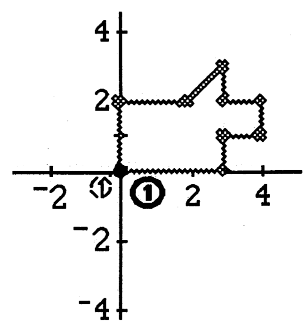
Valorie's transformation matrix

$$\overset{\text{old 1}}{[0\ \ 0]} \times \begin{bmatrix}1 & ^-1\\ 0 & 1\end{bmatrix} = \overset{\textbf{new 1}}{[0\ \ 0]}$$

$0 \times 1 + 0 \times 0 = 0$

$0 \times {}^-1 + 0 \times 1 = 0$

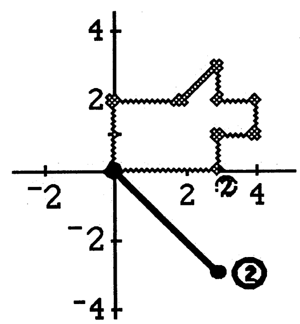
(the new points are maked with an • and a bold number in a bold circle)



$$\overset{\text{old2}}{[3\ \ 0]} \times \begin{bmatrix}1 & ^-1\\ 0 & 1\end{bmatrix} = \overset{\textbf{new2}}{[3\ \ ^-3]}$$

$3 \times 1 + 0 \times 0 = 3$

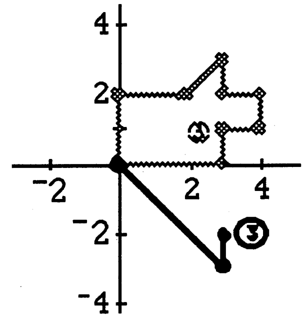
$3 \times {}^-1 + 0 \times 1 = {}^-3$



$$\overset{\text{old3}}{[3\ \ 1]} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \overset{\textbf{new3}}{[3\ \ {}^{-}2]}$$

$3 \times 1 + 1 \times 0 = 3$

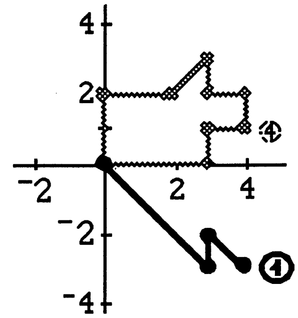
$3 \times {}^{-}1 + 1 \times 1 = {}^{-}2$



$$\overset{\text{old4}}{[4\ \ 1]} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \overset{\textbf{new4}}{[4\ \ {}^{-}3]}$$

$4 \times 1 + 1 \times 0 = 4$

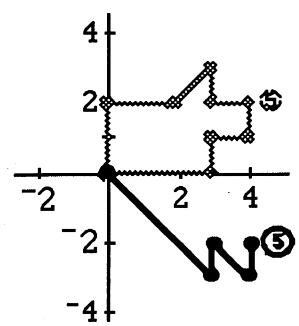
$4 \times {}^{-}1 + 1 \times 1 = {}^{-}3$



$$\overset{\text{old5}}{[4\ \ 2]} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \overset{\textbf{new5}}{[4\ \ {}^{-}2]}$$

$4 \times 1 + 2 \times 0 = 4$

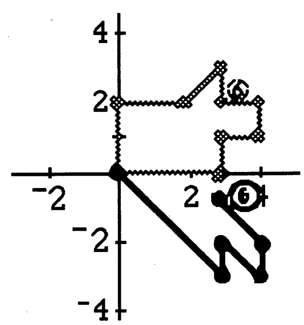
$4 \times {}^{-}1 + 2 \times 1 = {}^{-}2$



$$\overset{\text{old6}}{[3\ \ 2]} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \overset{\textbf{new6}}{[3\ \ {}^{-}1]}$$

$3 \times 1 + 2 \times 0 = 3$

$3 \times {}^{-}1 + 2 \times 1 = {}^{-}1$



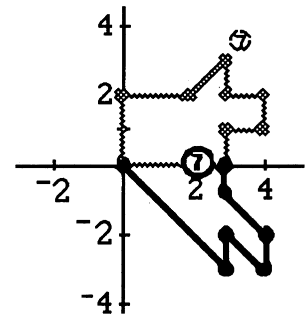
Valorie's transformation matrix

old7 **new7**

$$[3\ \ 3] \times \begin{bmatrix} 1 & {}^-1 \\ 0 & 1 \end{bmatrix} = [3\ \ 0]$$

$3 \times 1 + 3 \times 0 = 3$

$3 \times {}^-1 + 3 \times 1 = 0$

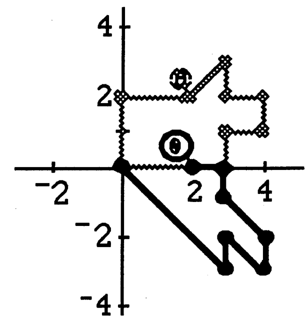


old8 **new8**

$$[2\ \ 2] \times \begin{bmatrix} 1 & {}^-1 \\ 0 & 1 \end{bmatrix} = [2\ \ 0]$$

$2 \times 1 + 2 \times 0 = 2$

$2 \times {}^-1 + 2 \times 1 = 0$

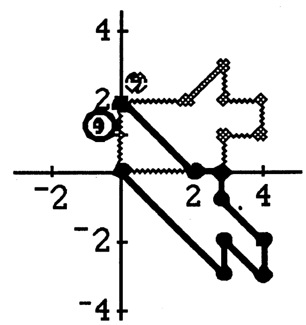


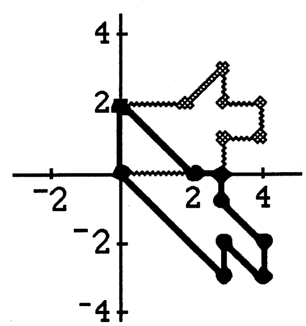
old9 **new9**

$$[0\ \ 2] \times \begin{bmatrix} 1 & {}^-1 \\ 0 & 1 \end{bmatrix} = [0\ \ 2]$$

$0 \times 1 + 2 \times 0 = 0$

$0 \times {}^-1 + 2 \times 1 = 2$





Valorie then drew the line from (0,2) to (0,0) to close the new doggie. What happened to the original doggie? What happens to horizontal lines? What happens to vertical lines? Are the lengths of line segments the same? Is the area within the shape the same?

3.1 What matrix would change this new shape back to the original?

Stop! Before going on, you pick a transformation matrix and follow the steps to transform your shape. Look for patterns. Try to figure out what your matrix does to the dog. Make up some questions about your work with matrices.

Ch. 4: Asking Questions & Other Student Work

Andreana's transformation matrix

① $\overset{O1}{\begin{bmatrix} 0 & 0 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N1}{\begin{bmatrix} 0 & 0 \end{bmatrix}}$

② $\overset{O2}{\cancel{\begin{bmatrix} 3 & 0 \end{bmatrix}}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N2}{\begin{bmatrix} 0 & 3 \end{bmatrix}}$

③ $\overset{O3}{\begin{bmatrix} 3 & 1 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N3}{\begin{bmatrix} -1 & 3 \end{bmatrix}}$

④ $\overset{O4}{\begin{bmatrix} 4 & 1 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N4}{\begin{bmatrix} -1 & 4 \end{bmatrix}}$

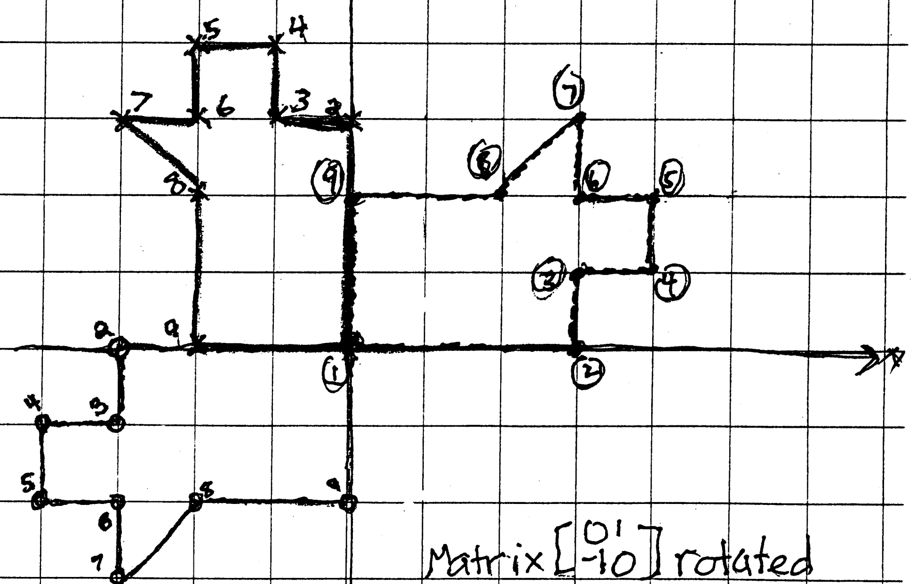
⑤ $\overset{O5}{\begin{bmatrix} 4 & 2 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N5}{\begin{bmatrix} -2 & 4 \end{bmatrix}}$

⑥ $\overset{O6}{\begin{bmatrix} 3 & 2 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N6}{\begin{bmatrix} -2 & 3 \end{bmatrix}}$

⑦ $\overset{O7}{\begin{bmatrix} 3 & 3 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N7}{\begin{bmatrix} -3 & 3 \end{bmatrix}}$

⑧ $\overset{O8}{\begin{bmatrix} 2 & 2 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N8}{\begin{bmatrix} -2 & 2 \end{bmatrix}}$

$\boxed{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$



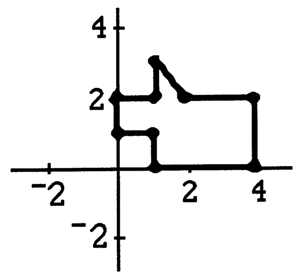
⑨ $\overset{O9}{\begin{bmatrix} 0 & 2 \end{bmatrix}} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \overset{N9}{\begin{bmatrix} -2 & 0 \end{bmatrix}}$

Matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ rotated the dog 90° counterclockwise.

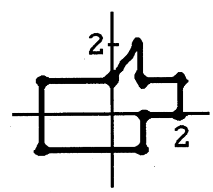
Matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ rotated the dog 180°

Andreana, 11 years old, made up the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; she realized that it rotated the dog 90° counterclockwise. I asked her to make up some questions about this situation. She brought in these questions her next session with pictures like those below:

1. How can you rotate the dog 180° ?
2. How do you make the dog bigger?
3. How do you make the dog smaller?
4. Is there any way to change the shape of the dog without squashing it?
5. Can you reverse the dog?



6. How can you put the dog in more than one quadrant?



She worked on her first question: How can you rotate the dog 180° ?

After 40 minutes of work, Andreana wrote, "To rotate the dog 180° I could rotate it 90° counterclockwise twice. I decided to try multiplying the matrices together before the coordinates."

beginning coordinates

$$\begin{bmatrix} 4 & 1 \end{bmatrix} \times \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{} =$$

$$= \begin{bmatrix} 4 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -1 \end{bmatrix}$$

ending coordinates

The result was the same when I first multiplied the coordinates by the 2x2 matrix, and then multiplied the result by the second, in this case the same, 2x2 matrix."

beginning coordinates

$$\begin{bmatrix} 4 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \end{bmatrix}$$

ending coordinates

From her work above, there were two things Andreana and I talked about, one, that the multiplication of matrices is associative; and two, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ acts like the number -1, and gives the opposite of each number in the pair of coordinates.

We also talked about rotating the 90° ccw twice and realized $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ which is the matrix which rotates 180°. Then $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which rotates the dog 270° ccw or 90° cw.

① $\overset{O1}{\begin{bmatrix} 0 & 0 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N1}{\begin{bmatrix} 0 & 0 \end{bmatrix}}$

② $\overset{O2}{\begin{bmatrix} 3 & 0 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N2}{\begin{bmatrix} 0 & -3 \end{bmatrix}}$

③ $\overset{O3}{\begin{bmatrix} 3 & 1 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N3}{\begin{bmatrix} 1 & -3 \end{bmatrix}}$

④ $\overset{O4}{\begin{bmatrix} 4 & 1 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N4}{\begin{bmatrix} 1 & -4 \end{bmatrix}}$

⑤ $\overset{O5}{\begin{bmatrix} 4 & 2 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N5}{\begin{bmatrix} 2 & -4 \end{bmatrix}}$

⑥ $\overset{O6}{\begin{bmatrix} 3 & 2 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N6}{\begin{bmatrix} 2 & -3 \end{bmatrix}}$

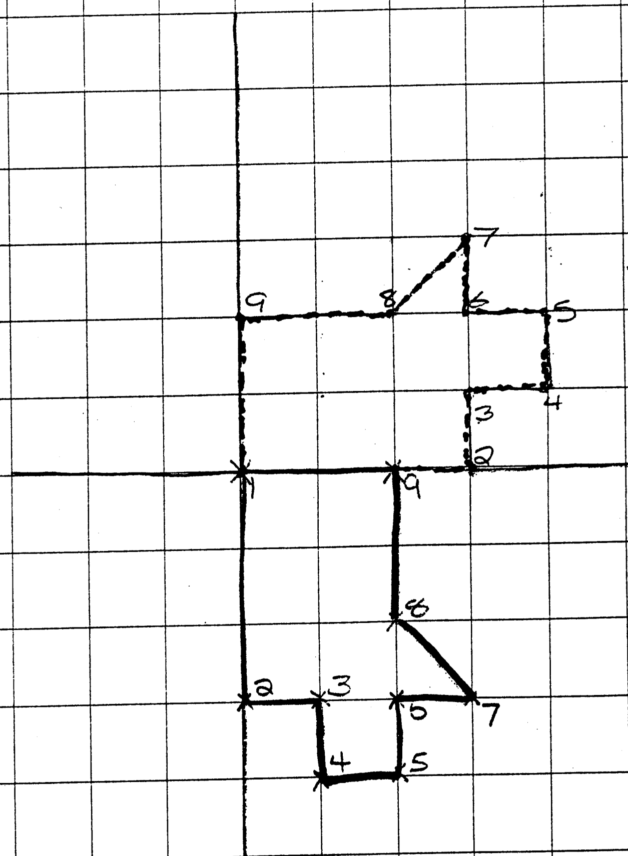
⑦ $\overset{O7}{\begin{bmatrix} 3 & 3 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N7}{\begin{bmatrix} 3 & -3 \end{bmatrix}}$

⑧ $\overset{O8}{\begin{bmatrix} 2 & 2 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N8}{\begin{bmatrix} 2 & -2 \end{bmatrix}}$

⑨ $\overset{O9}{\begin{bmatrix} 0 & 2 \end{bmatrix}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \overset{N9}{\begin{bmatrix} 2 & 0 \end{bmatrix}}$

I predict that this matrix will rotate the dog 90° clockwise

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotated the dog 90° clockwise

① $\overset{O1}{[0\ 0]} \times \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \overset{N1}{[0\ 0]}$

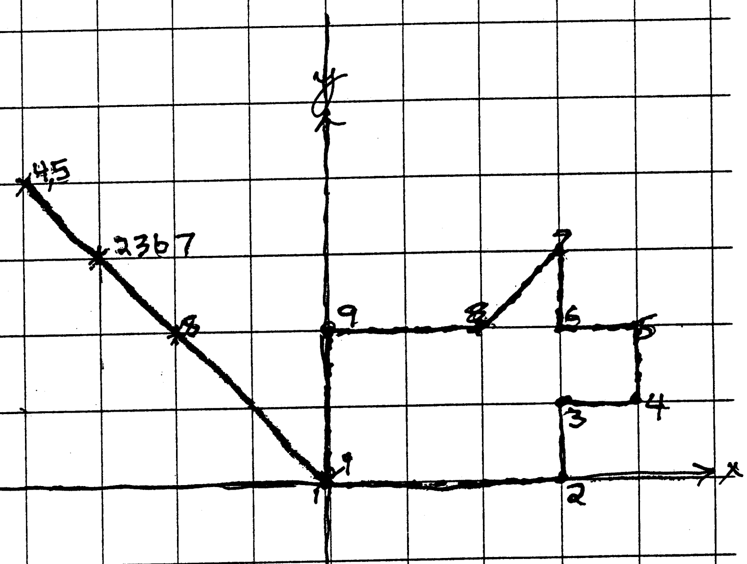
② $\overset{O2}{[3\ 0]} \times \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \overset{N2}{[-3\ 3]}$

andrea

③ $\overset{O3}{[3\ 1]} \times \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \overset{N3}{[-3\ 3]}$

④ $\overset{O4}{[4\ 1]} \times \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \overset{N4}{[-4\ 4]}$

⑤ $\overset{O5}{[4\ 2]} \times \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \overset{N5}{[-4\ 4]}$



Any matrix $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ will squish the dog.

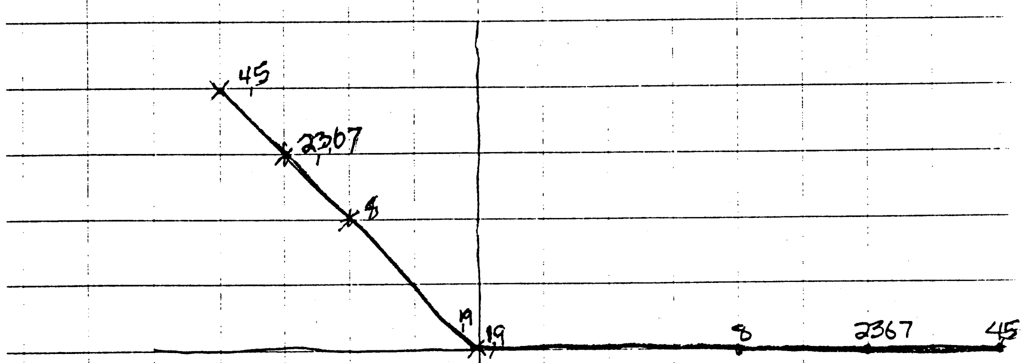
Matrix $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ squashes dog.

Andreana made up the matrix $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ which "squashed" the dog along the y = ⁻x line. She also made a picture of how the points went to the new points. All the points on the vertical line x=3 for example, went to the point (⁻3,3)

We talked about what matrix would "unsquish" the dog. She tried $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ which didn't work, but low and behold she came up with the same matrix she started with.

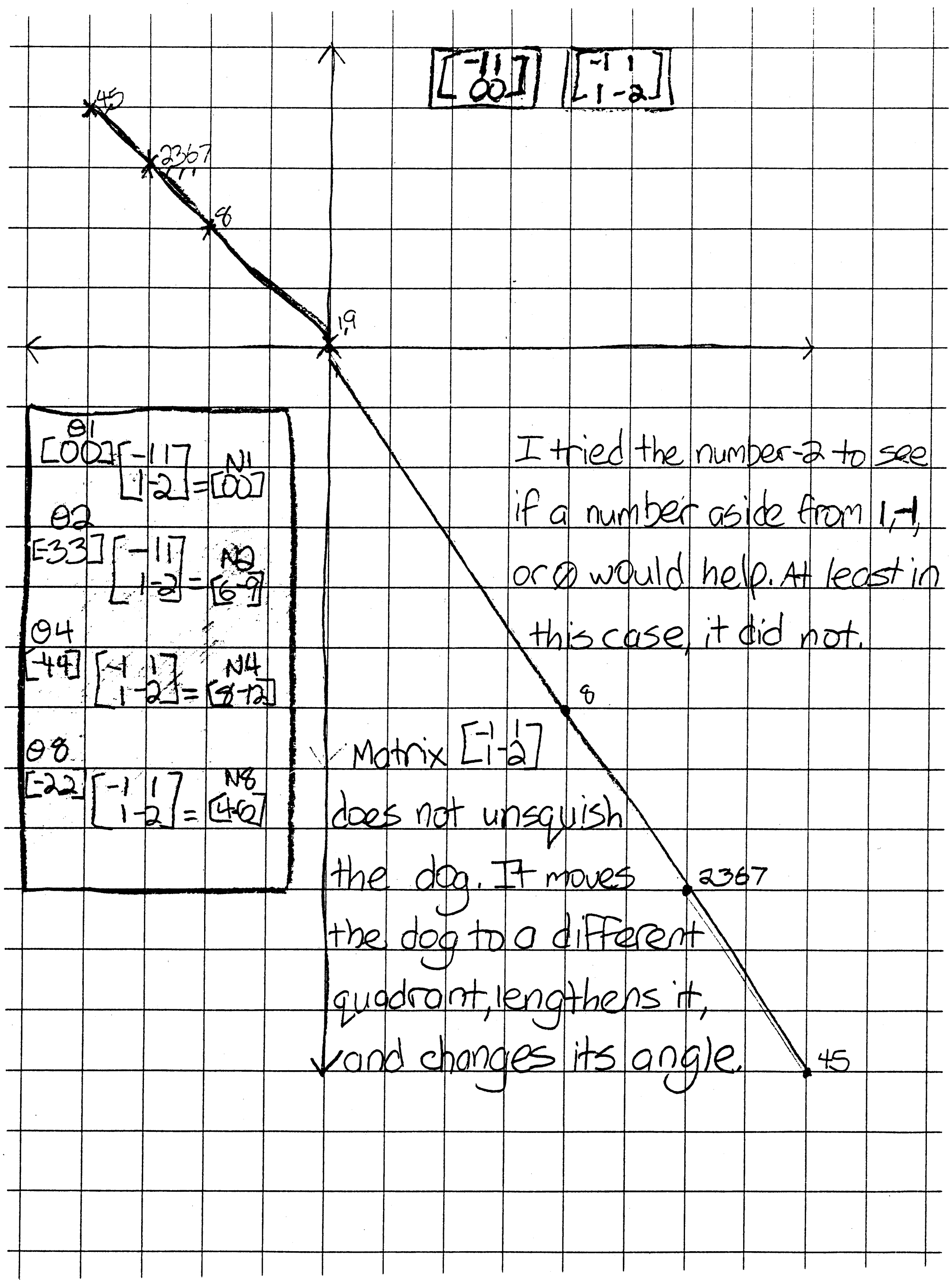
$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. The matrix $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ acts like the "1" matrix, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Of course it doesn't always work that way.

$\boxed{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}}$ $\boxed{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}$



$\boxed{\begin{array}{l} ①\ [0\ 0]\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = [0\ 0] \\ ③\ [-3\ 3]\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = [6\ 0] \\ ④\ [-4\ 4]\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = [8\ 0] \\ ⑧\ [-2\ 2]\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = [4\ 0] \end{array}}$

Matrix $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ does not unsquish the dog. It places it against the x-axis and lengthens it.



I tried the number -2 to see if a number aside from 1, -1, or 0 would help. At least in this case, it did not.

Matrix $\begin{bmatrix}-1 & 1\\ 1 & -2\end{bmatrix}$ does not unsquish the dog. It moves the dog to a different quadrant, lengthens it, and changes its angle.

andreana

$\boxed{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}}$

$O1$ ① $\begin{bmatrix}0 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \overset{N1}{\begin{bmatrix}0 & 0\end{bmatrix}}$

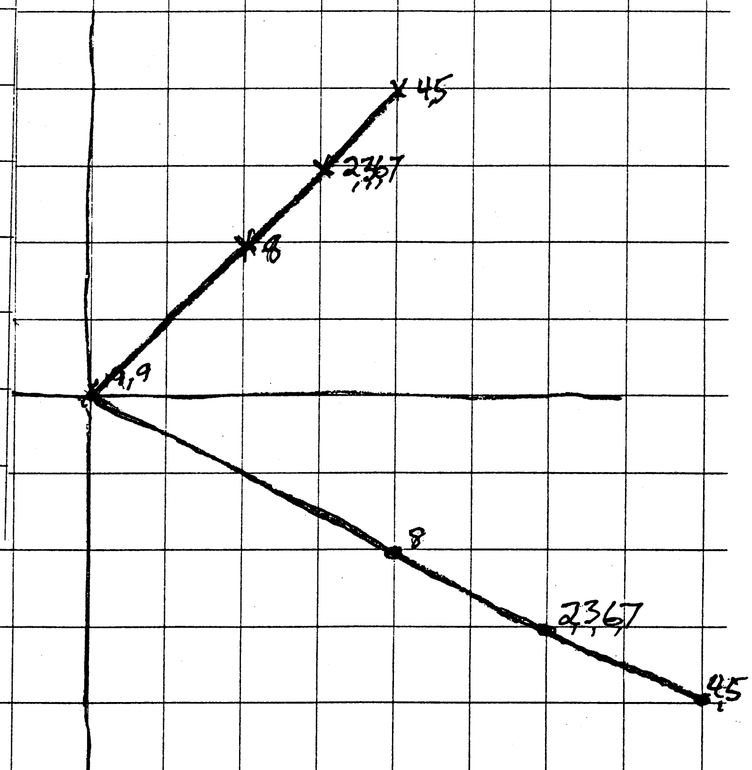
$O2$ ② $\begin{bmatrix}3 & 3\end{bmatrix} \cdot \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \overset{N2}{\begin{bmatrix}6 & -3\end{bmatrix}}$

$O3$ ③ $\begin{bmatrix}3 & 3\end{bmatrix} \cdot \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \overset{N3}{\begin{bmatrix}6 & -3\end{bmatrix}}$

$O4$ ④ $\begin{bmatrix}4 & 4\end{bmatrix} \cdot \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \overset{N4}{\begin{bmatrix}8 & -4\end{bmatrix}}$

$O5$ ⑤ $\begin{bmatrix}4 & 4\end{bmatrix} \cdot \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \overset{N5}{\begin{bmatrix}8 & -4\end{bmatrix}}$

$O8$ ⑧ $\begin{bmatrix}2 & 2\end{bmatrix} \cdot \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \overset{N8}{\begin{bmatrix}4 & -2\end{bmatrix}}$



Matrix $\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$ does not unsquish the dog. It puts it in another quadrant, at another angle (about 27 degrees). It also lengthens the line.

There is more thon one shape that will make the Y shape when the $\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}$ matrix is applied to it.

O1 $[0\ 0]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [0\ 0]$ N1

O2 $[3\ 3]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [-3\ 3]$ N2

O3 $[3\ 2]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [-3\ 3]$ N3

O4 $[4\ 4]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [-4\ 4]$ N4

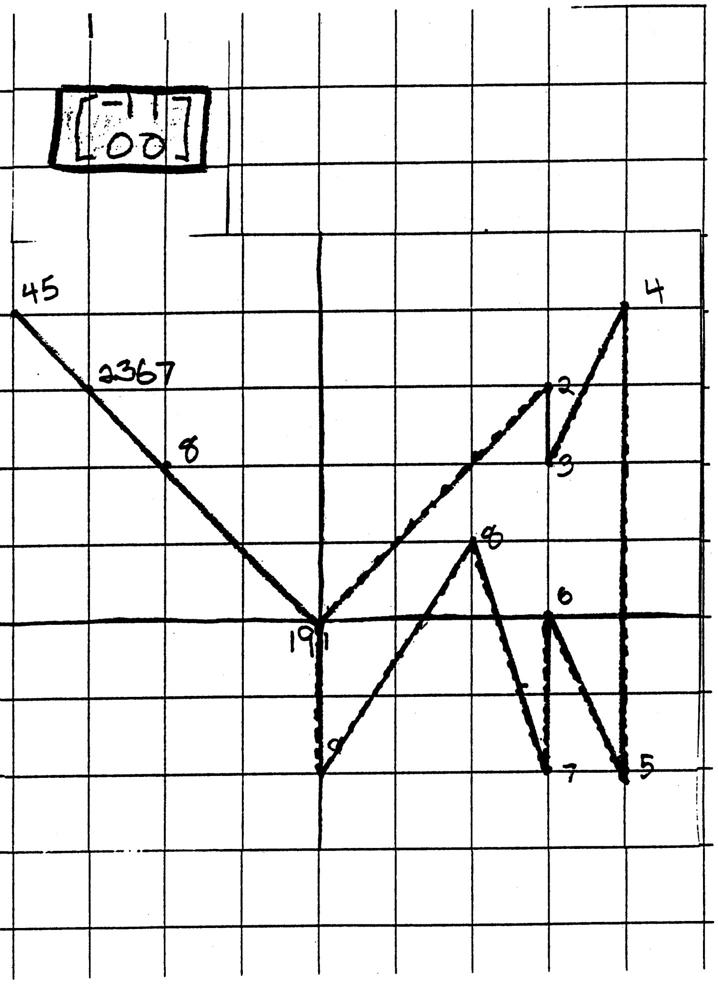
O5 $[4\ -2]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [-4\ 4]$ N5

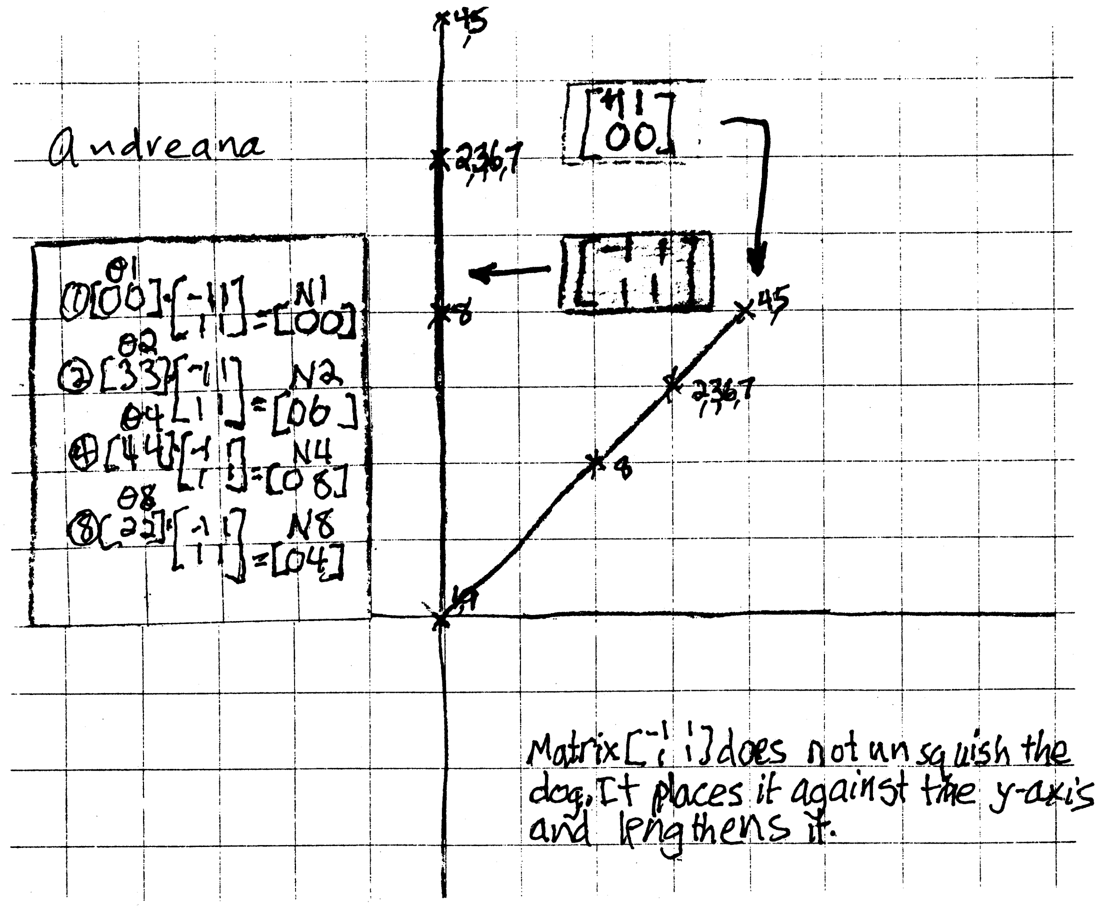
O6 $[3\ 0]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [-3\ 3]$ N6

O7 $[3\ -2]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [-3\ 3]$ N7

O8 $[2\ 1]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [-2\ 2]$ N8

O9 $[0\ 2]\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix} = [0\ 0]$ N9





In order to unsquish the dog, Andreana tried to find a matrix, which when multiplied by the squishing matrix, would give the "1" matrix. She soon realized there would be 81 matrices to try! She also realized there wasn't a matrix that works, because there was no way to get a 1 in the second row, second column of the answer, when the second row of the squishing matrix has two zeros. She raised the question - how do you find the reciprocal of a (2x2) matrix? We didn't pursue this.

The question came up: is there a minimal list of matrices from which we can get the rest? We didn't attempt to answer this, however.

Andreana answered her questions 2 and 4 on the next page. At the time of printing Andreana was trying to translate the dog (question 6), and was finding patterns to the points as they are squished (see Will's diagram).

① $\overset{O1}{[0\ 0]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\overset{N1}{[0\ 0]}$

② $\overset{O2}{[3\ 0]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\overset{N2}{[3\ -3]}$

③ $\overset{O3}{[3\ 1]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\overset{N3}{[4\ -3]}$

④ $\overset{O4}{[4\ 1]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\overset{N4}{[5\ -4]}$

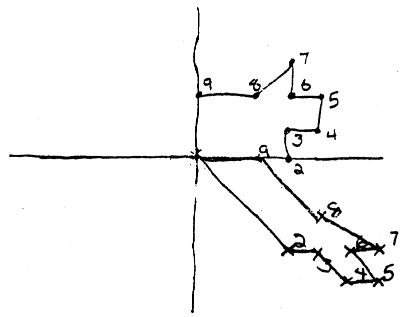
⑤ $\overset{O5}{[4\ 2]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\overset{N5}{[6\ -4]}$

⑥ $\overset{O6}{[3\ 2]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\overset{N6}{[5\ -3]}$

⑦ $\overset{O7}{[3\ 3]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\overset{N7}{[6\ -3]}$

⑧ $\overset{O8}{[2\ 2]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}\overset{N8}{[4\ -2]}$

⑨ $\overset{O9}{[0\ 2]}\begin{bmatrix}1&-1\\1&0\end{bmatrix}\overset{N9}{[2\ 0]}$



Matrix $\begin{bmatrix}1&-1\\1&0\end{bmatrix}$ changes shape of dog.

---

① $\overset{O1}{[0\ 0]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N1}{[0\ 0]}$

② $\overset{O2}{[3\ 0]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N2}{[6\ 0]}$

③ $\overset{O3}{[3\ 1]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N3}{[6\ 2]}$

④ $\overset{O4}{[4\ 1]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N4}{[8\ 2]}$

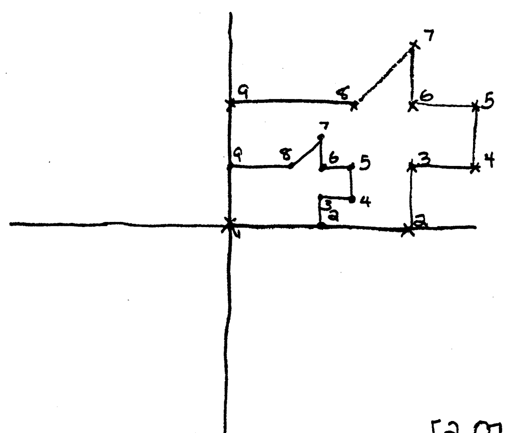
⑤ $\overset{O5}{[4\ 2]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N5}{[8\ 4]}$

⑥ $\overset{O6}{[3\ 2]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N6}{[6\ 4]}$

⑦ $\overset{O7}{[3\ 3]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N7}{[6\ 6]}$

⑧ $\overset{O8}{[2\ 2]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N8}{[4\ 4]}$

⑨ $\overset{O9}{[0\ 3]}\begin{bmatrix}2&0\\0&2\end{bmatrix}=\overset{N9}{[0\ 6]}$



Matrix $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ doubles size of dog.

Matrix $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ is matrix $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ multiplied by 2.

Mark Age 7

Mark's Transformation Matrix

old 5 new 5

$$[4\ 2] \times \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = [2\ 2]$$

old 8 new 8

$$(2\ 2) \times \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = (2\ 0)$$

old 1 new 1

$$[0\ 0] \times \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = (0\ 0)$$

old 9 new 9

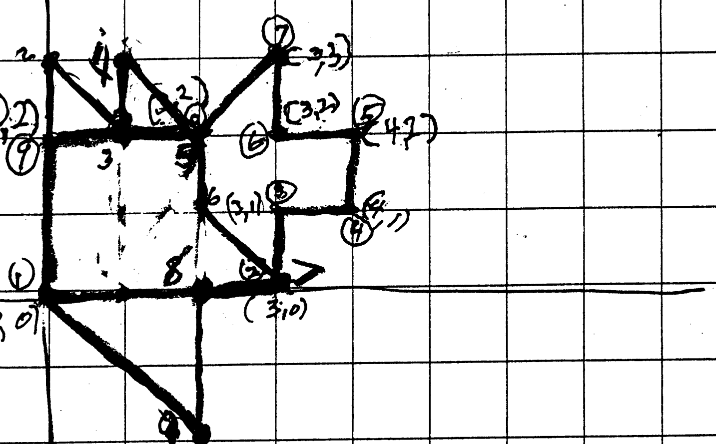
$$(0\ 2) \times \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = (2\ {-2})$$

old 2 new 2

$$[3\ 0] \times \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = (0\ 3)$$

old 3 new 3

$$(3\ 0) \times \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = (1\ 2)$$



old 4 new 4

$$(4\ 1) \times \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = (1\ 3)$$

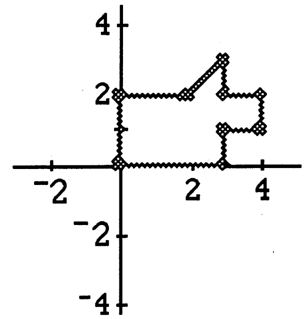
old 6 new 6

$$(3\ 2) \times \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = (2\ 1)$$

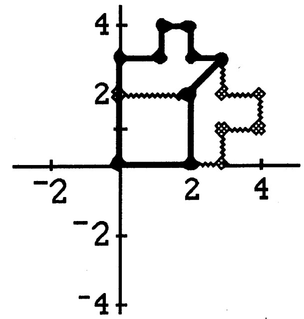
old 7 new 7

$$(3\ 3) \times \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = (3\ 0)$$

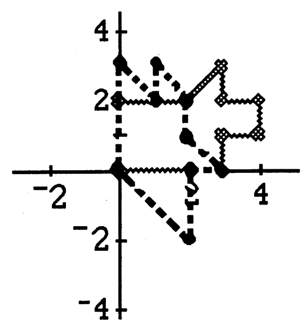
Showing #29 (refection in y=x line) x #22 (shear down) = Mark's #30



#29 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ takes the original dog to its reflection in the y=x line

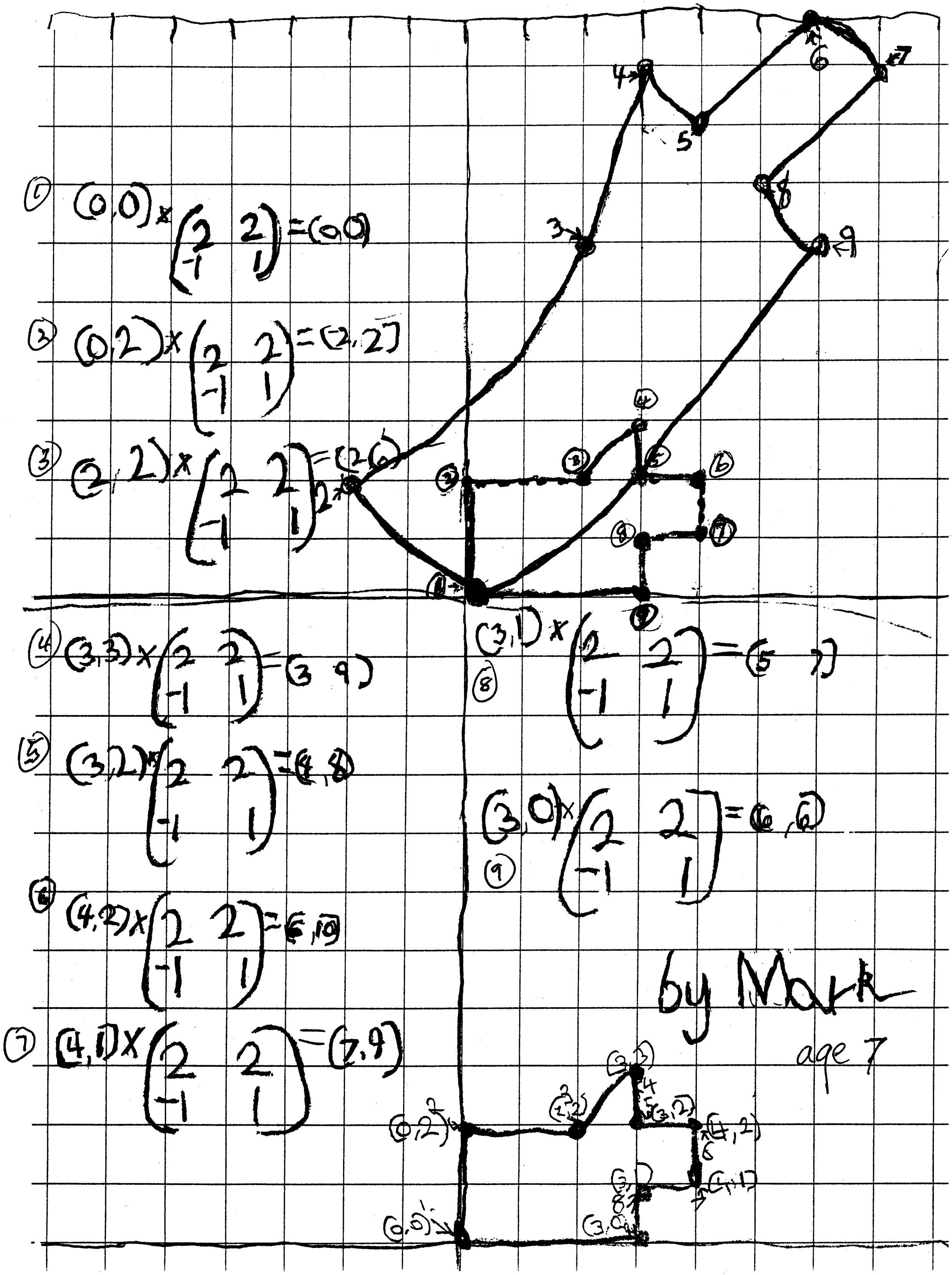


#22 $\begin{bmatrix} 1 & {}^{-}1 \\ 0 & 1 \end{bmatrix}$ takes the refection in the y=x line and shears it down to get the same thing as Mark's did with #30

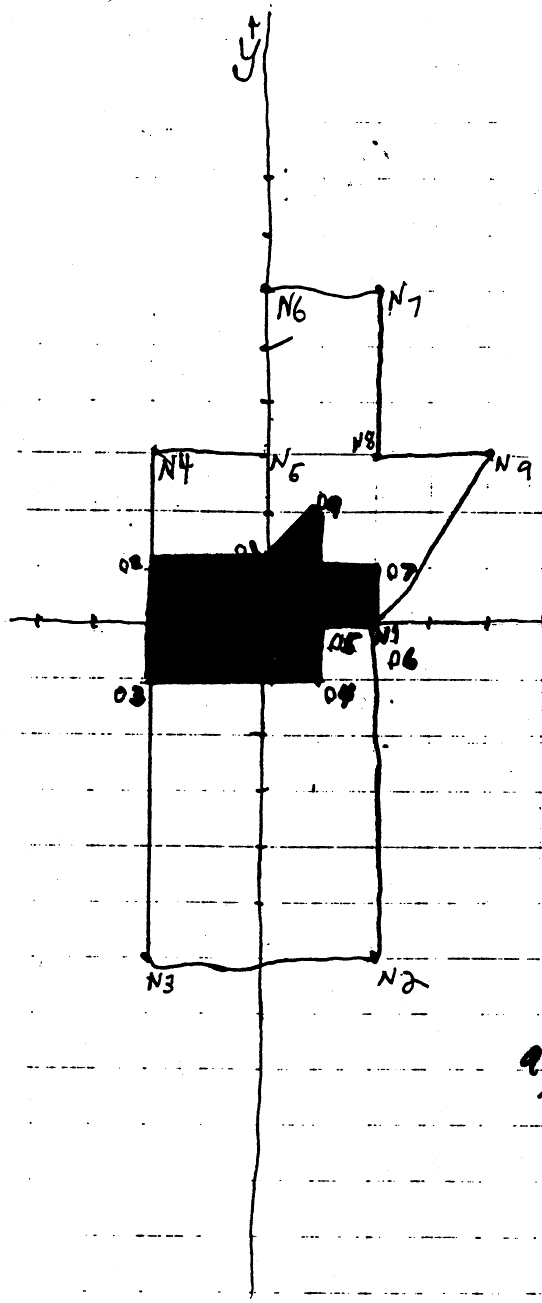


So #29 x #22 = #30

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & {}^{-}1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & {}^{-}1 \end{bmatrix}$$



① (0,0) x (2 2 / -1 1) = (0,0)
② (0,2) x (2 2 / -1 1) = (2,2]
③ (2,2) x (2 2 / -1 1) = (2,6)
④ (3,3) x (2 2 / -1 1) = (3 9)
⑤ (3,2) x (2 2 / -1 1) = (4,8)
⑥ (4,2) x (2 2 / -1 1) = (6,10)
⑦ (4,1) x (2 2 / -1 1) = (7,9)
⑧ (3,1) x (2 2 / -1 1) = (5 7]
⑨ (3,0) x (2 2 / -1 1) = (6,6)
by Mark
age 7



O1 $(0,1)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(2,0)$ N1

$(-2,1)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(2,-6)$

$(-2,-1)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(-2,-6)$

$(1,-1)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(-2\ 3)$

$(1,0)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(0,3)$

$(2,0)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(0,6)$

$(2,1)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(2,6)$

$(1,1)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(2,3)$

$(1,2)\begin{pmatrix}0&3\\2&0\end{pmatrix}=(4,3)$

age 7 Matt,

8/3/87

prediction for same but no flip

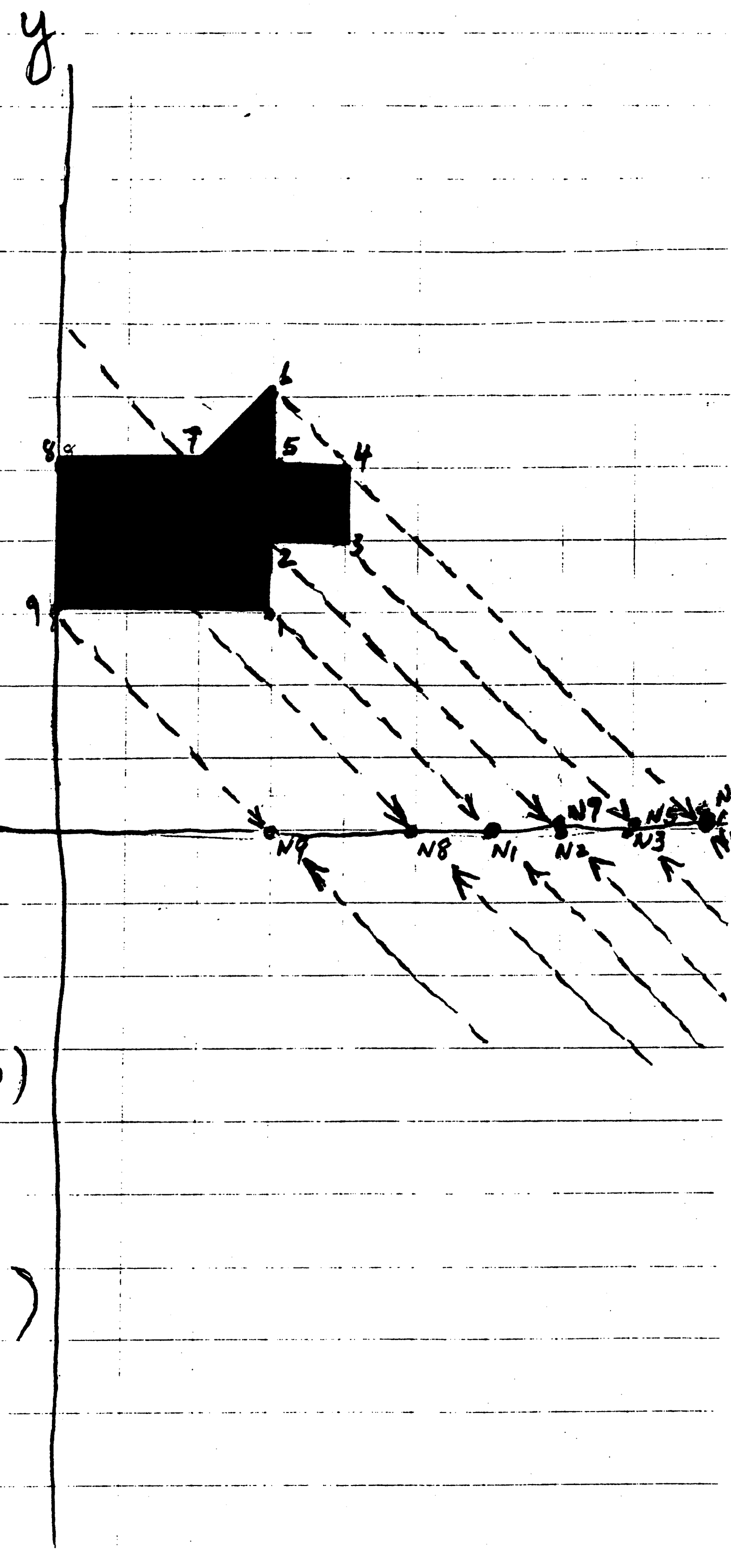
$\begin{pmatrix}1&3\\2&1\end{pmatrix}$

3x ↕

2x ↔ without flip

Will, 9th gr.

$(x,y) \to (x+y, 0)$



$(3\ 3)\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = (6\ 0)$

$(3\ 4)\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = (7\ 0)$

$(4\ 4)\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = (8\ 0)$

SOME ALGEBRA AND TRANSFORMATIONS WITH MATRICES

by Theodore Gray[1]

THE ALGEBRA

I read about "candy-store arithmetic" in "Explorations in Mathematics" Chapter 37[2]. Don gave me these to multiply

$$(2\ 3\ 5) \times \begin{pmatrix}1\\7\\9\end{pmatrix} = (56). \qquad (2\times 1) + (3\times 7) + (5\times 9)$$

Then he gave me $\begin{pmatrix}3&5\\7&9\end{pmatrix}\times\begin{pmatrix}2&4\\6&8\end{pmatrix}$; at first I got $\begin{pmatrix}26&58\\50&114\end{pmatrix}$, but Don showed me I wasn't using the same scheme $\begin{pmatrix}3&5\\7&9\end{pmatrix}\times\begin{pmatrix}2&4\\6&8\end{pmatrix}$. At that point I got $\begin{pmatrix}36&52\\68&100\end{pmatrix}$, the right answer. I generalized the multiplication of 2-by-2 matrices like this:

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}\times\begin{pmatrix}E&F\\G&H\end{pmatrix}=\begin{pmatrix}AE+BG&AF+BH\\CE+DG&CF+DH\end{pmatrix}$$

Next I added $\begin{pmatrix}3&8\\4&5\end{pmatrix}+\begin{pmatrix}6&2\\7&9\end{pmatrix}=\begin{pmatrix}9&10\\11&14\end{pmatrix}$; then generalized $\begin{pmatrix}A&B\\C&D\end{pmatrix}+\begin{pmatrix}E&F\\G&H\end{pmatrix}=\begin{pmatrix}A+E&B+F\\C+G&D+H\end{pmatrix}$

Don asked me to find a 2-by-2 matrix that acts like 1. In other words, what matrix goes here

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}\times\boxed{}=\begin{pmatrix}A&B\\C&D\end{pmatrix}$$

to make this true. I came up with $\begin{pmatrix}1&1\\1&1\end{pmatrix}$

but it didn't work; instead it did this: $\begin{pmatrix}A&B\\C&D\end{pmatrix}\times\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}A+B&A+B\\C+D&C+D\end{pmatrix}$

My dad helped me a bit and I found that $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ worked. The following page shows some matrices I found and generalized.

[1] Theo, 11 years old, did this work during a 2-week summer session in THE MATH PROGRAM; Don Cohen was his teacher and assisted him in putting it all together.

[2] "Explorations in Mathematics"; R. B. Davis, Addison-WEsley; Mass.

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}=\begin{pmatrix}-1 & -4\\-3 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}=\begin{pmatrix}2 & 8\\6 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}=\begin{pmatrix}3 & 12\\9 & 6\end{pmatrix}$$

$$\begin{pmatrix}A & B\\C & D\end{pmatrix}\times\begin{pmatrix}E & 0\\0 & E\end{pmatrix}=\begin{pmatrix}AE & BE\\CE & DE\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}=\begin{pmatrix}-4 & -1\\-3 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}=\begin{pmatrix}4 & 1\\2 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}0 & 2\\2 & 0\end{pmatrix}=\begin{pmatrix}8 & 2\\4 & 6\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}0 & 3\\3 & 0\end{pmatrix}=\begin{pmatrix}12 & 3\\6 & 9\end{pmatrix}$$

$$\begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{pmatrix}0 & E\\E & 0\end{pmatrix}=\begin{pmatrix}BE & AE\\DE & CE\end{pmatrix}$$

$$\begin{pmatrix}A & B\\C & D\end{pmatrix}\times\begin{pmatrix}E & E\\0 & 0\end{pmatrix}=\begin{pmatrix}AE & AE\\CE & CE\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0 & 0\\-1 & -1\end{pmatrix}=\begin{pmatrix}-4 & -4\\-2 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}=\begin{pmatrix}4 & 4\\2 & 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0 & 0\\2 & 2\end{pmatrix}=\begin{pmatrix}8 & 8\\4 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0 & 0\\3 & 3\end{pmatrix}=\begin{pmatrix}12 & 12\\6 & 6\end{pmatrix}$$

$$\begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{pmatrix}0 & 0\\E & E\end{pmatrix}=\begin{pmatrix}BE & BE\\DE & DE\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}-1 & -1\\0 & 0\end{pmatrix}=\begin{pmatrix}-1 & -1\\-3 & -3\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}=\begin{pmatrix}1 & 1\\3 & 3\end{pmatrix}$$

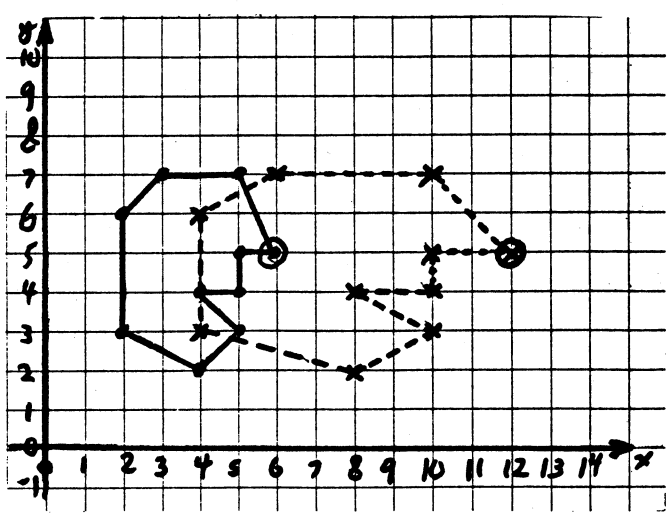
$$\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\times\begin{pmatrix}2 & 2\\0 & 0\end{pmatrix}=\begin{pmatrix}2 & 2\\6 & 6\end{pmatrix}$$

I then checked to see if the following works:

		WORKED?
Commutative Law for Addition	$A + B = B + A$	yes
Commutative Law for Multiplication	$A \times B = B \times A$	no
Distributive Law	$A \times (B + C) = (A \times B) + (A \times C)$	yes
Associative Law for Addition	$A + (B + C) = (A + B) + C$	yes
Associative Law for Multiplication	$A \times (B \times C) = (A \times B) \times C$	yes

THE GEOMETRY

Here is an example of a transformation: $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ will be the transformation matrix



$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

old point | new point

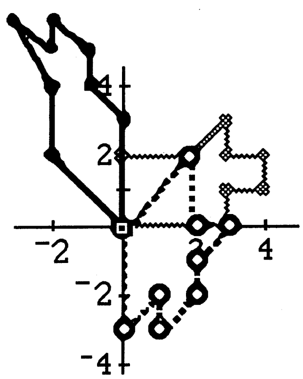
stretching matrix

old x	old y	new x	new y
4	2	8	2
2	3	4	3
2	6	4	6
3	7	6	7
5	7	10	7
5	5	10	5
6	5	12	5
5	4	10	4
4	4	8	4
5	3	10	3

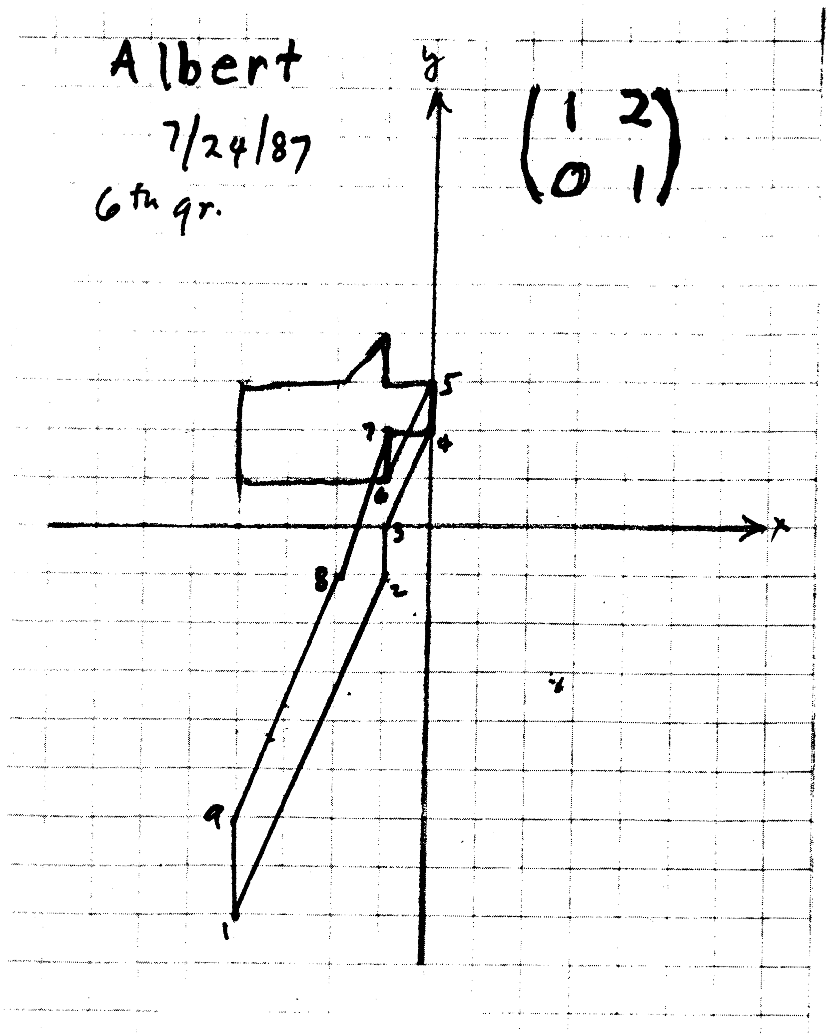
I took other 2-by-2 matrices to enlarge, flatten, rotate 90° and reflect.

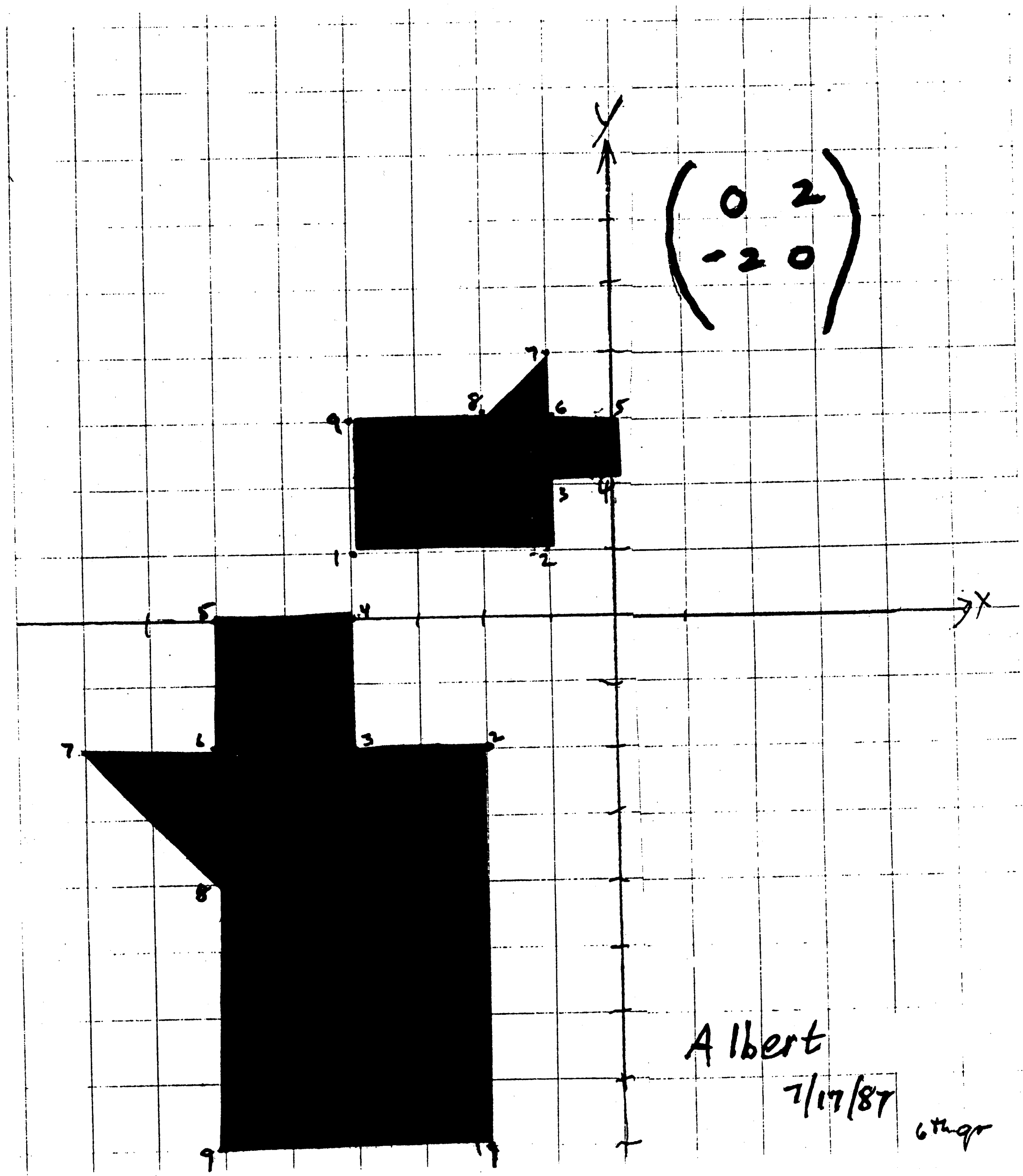
Theo used a column matrix for the point coordinates $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$, which I showed him twenty years ago, instead of the row matrix I have been using recently, like $\begin{bmatrix} 6 & 5 \end{bmatrix}$. The result is the same using either the column matrix or the row matrix for his matrix above:

$$\begin{bmatrix} 6 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 5 \end{bmatrix}$$



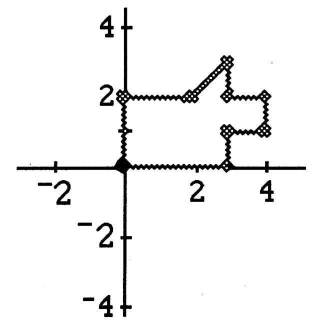
If we take a different transformation matrix, however, like $\begin{bmatrix} 0 & ^-1 \\ 1 & 1 \end{bmatrix}$, the resulting points will not be the same! The diagram on the left shows the original dog, the circle-dashed dog is the result of the row matrix for the coordinates as we have been doing it, and the heavy-lined dog is the result of doing Theo's column matrix for coordinates. The dashed dog is the result of a 90° clockwise rotation and a shear up. The heavy-lined dog is the result of a 90° counterclockwise rotation and a shear up. So the two methods are related after all!


Albert
7/24/87
6th gr.
y
x
$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
1
2
3
4
5
6
7
8
9

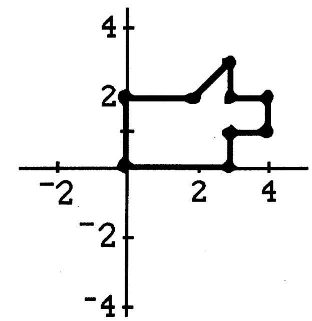

y
x
$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$
Albert
7/17/87
6th gr

Ch. 5: Some special matrices and their resulting transformations

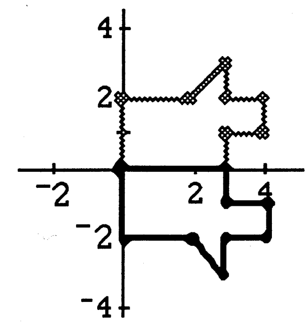
The "**zero**" matrix, corresponds to 0 in the regular numbers; that is, when you add another 2x2 matrix to it (which Theo did), you end up with the same one as you started with. This is # 41 on the list of 81: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$



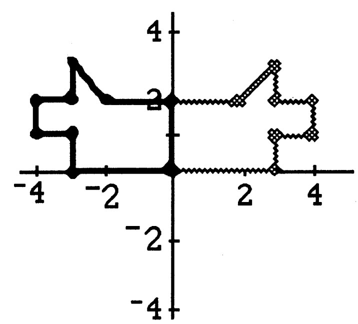
The "**1**" matrix corresponds to 1 in the regular numbers; that is, when you multiply another 2x2 matrix by it you end up with the same one as you started with: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, this is # 13 on list of 81.

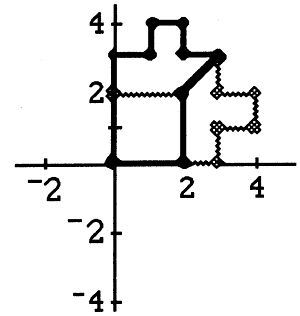


This matrix **reflects** the shape about the **x**-axis: $\begin{bmatrix} 1 & 0 \\ 0 & ^{-}1 \end{bmatrix}$. It is # 15 on list of 81.



This matrix **reflects** the shape about **y**-axis. It is #67 on list of 81: $\begin{bmatrix} ^{-}1 & 0 \\ 0 & 1 \end{bmatrix}$



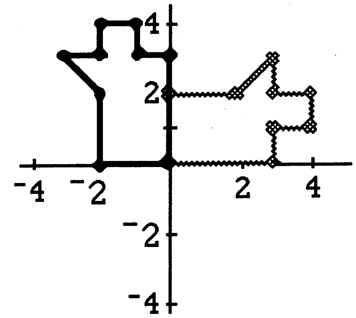


This matrix **reflects** the shape about the **y=x** line:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In effect it interchanges the x and y coordinates. It's like finding the inverse of a function! It is # 29 on list of 81.

Three **rotations** are used in the "Flight Simulator":

1. the airelons (the trailing edge of the wing) control the rotation of the plane about its longitudinal axis, allowing for banking or rolling.
2. the elevators on the rear wings of the plane control the rotation about its lateral axis and moving the nose up or down.
3. the rudder controls the turning of the plane about its vertical axis (yaw)- it turns the plane left or right.

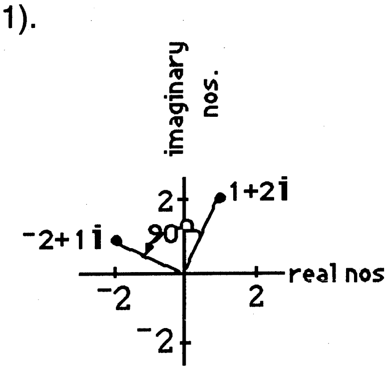


The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ rotates the shape 90° counter-clockwise (Andreana's matrix). This is #35 on the list of 81. We had a matrix above that acts like 1, this matrix acts like the imaginary number **i**. Below is an example in which the point (1,2) in the complex plane, is multiplied by **i** in the form of the matrix above resulting in the point ($^-2$,1), which is the original point rotated 90° ccw. Using numbers, **i** × (1 + 2**i**) = 1**i** + 2**i**^{2} = 1**i** + $^-2$ = $^-2$ + 1**i** (since **i**^{2} = $^-1$).

$^-2+1\mathbf{i}$

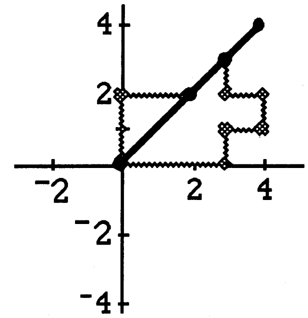
$$[1 \;\; 2] \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [-2 \;\; 1]$$

1+2**i**

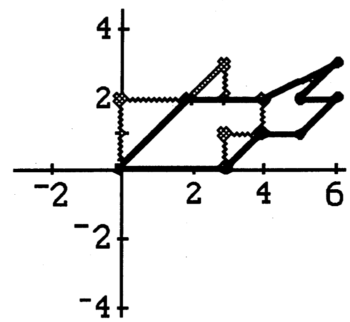


5.1 What matrix will rotate the figure 90° clockwise? **5.2** 60° counterclockwise?
5.3 What happens if you multiply this matrix above by itself ?

"**Squashing**" matrices: These matrices take the points on the dog and transform them to points on a line. The transformation on the right was done with the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$; it is #5 on the list of 81. You might like to draw lines from the original points to the new points to see how the transformation happens. These flattening matrices do different things. (See Will's and Andreana's matrices in Ch. 4).

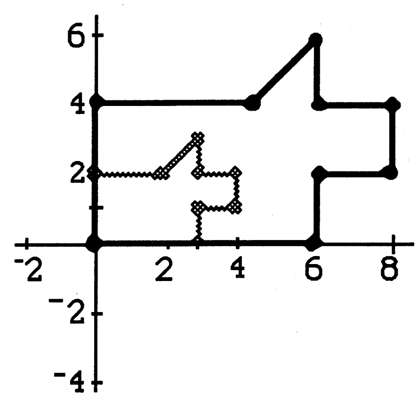


The **shear** transformation like the one on the right, is similar to the one in the fish diagrams from D'Arcy Thompson as shown on the cover, but this one is at an angle of 45°: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This is # 10 matrix on the list of 81. A shear happens when stresses to an object occur, like in metal or to animal fossils in rock, or as the shape of fish differs from one kind to another. Valerie's matrix results in a shear down. Why? What matrix will cause the dog to be sheared up? Left?



The **scaling** matrix:

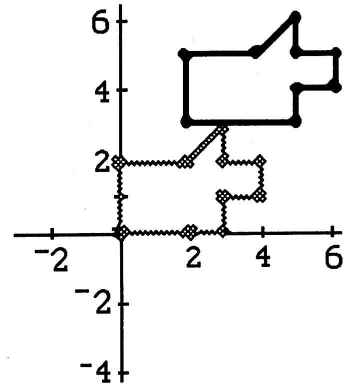
The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ doubles each of the coordinates and is called a **scaling**, or enlargement, in this case.



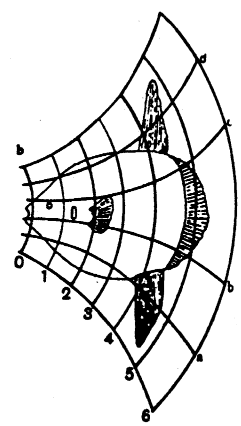
5.4 How does the area of the new shape compare to the area of the old shape?
How do the perimeters compare?
How would you stretch the dog to the right, but not up?

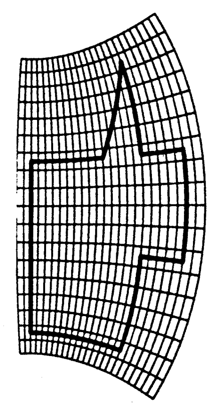
In order to **translate** the dog we need to either <u>not</u> use matrices as in appendix 2 or use 3x3 matrices. If we use a 1x3 matrix for the point instead of 1x2, then use a 3x3 matrix as the transformation matrix, we get a translation. The 1 in the third column of the first matrix is not used as is the 1 in the third row and third column of the second matrix. The 2 and 3 in the third row will move the dog 2 units to the right and 3 units up to get the new dog.

$$[0 \ 0 \ 1] \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = [2 \ \ 3 \ \ 1]$$



I struggled with a BASIC program (appendix 4) to get the **non-linear transformation** of the fish shown on page 131 of D'Arcy Thompson's book.





Theo was able change the dog using *Mathematica*, with the transformation

NewX = sinh(OldX) * cos(a*OldY)
NewY = cosh(OldX) * sin(a*OldY)

as shown in McGregor and Watt, page137, where

$$\sinh(x) = \text{hyperbolic } \sin(x) = \frac{e^{x} - e^{-x}}{2} =$$

for the computer, (EXP(x) - EXP(-x))/2 and

$$\cosh(x) = \text{hyperbolic } \cos(x) = \frac{e^{x} + e^{-x}}{2} =$$

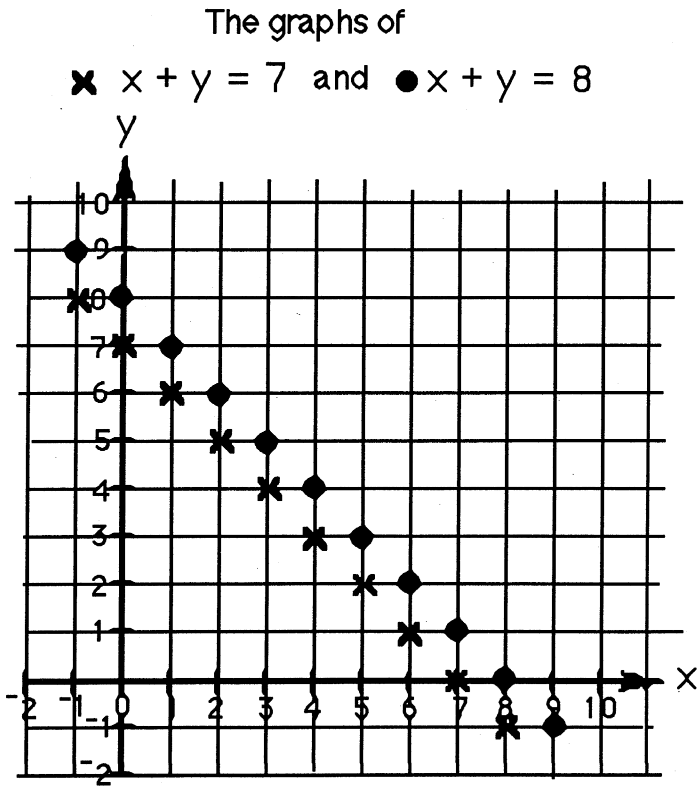
for the computer, (EXP(x) + EXP(-x))/2

The programs are in appendix 4.

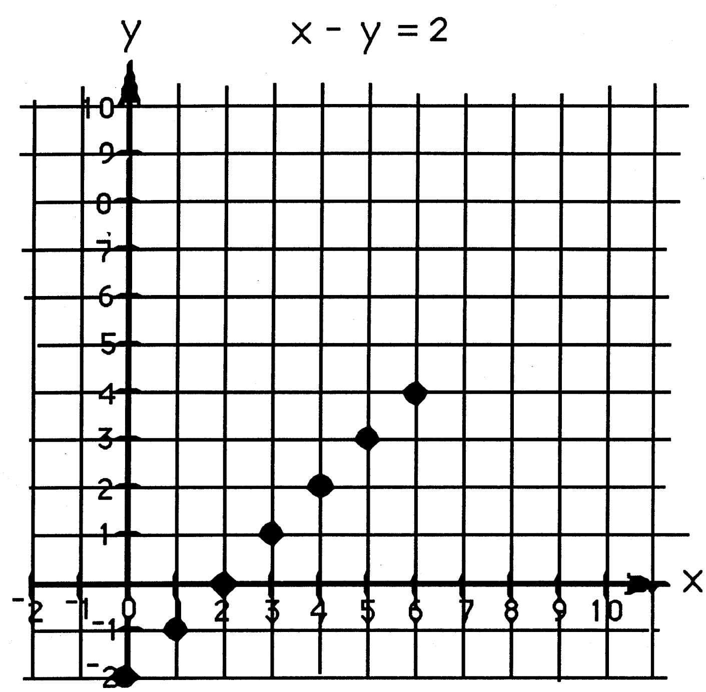
Appendix 1: Selected answers

Ch. 1

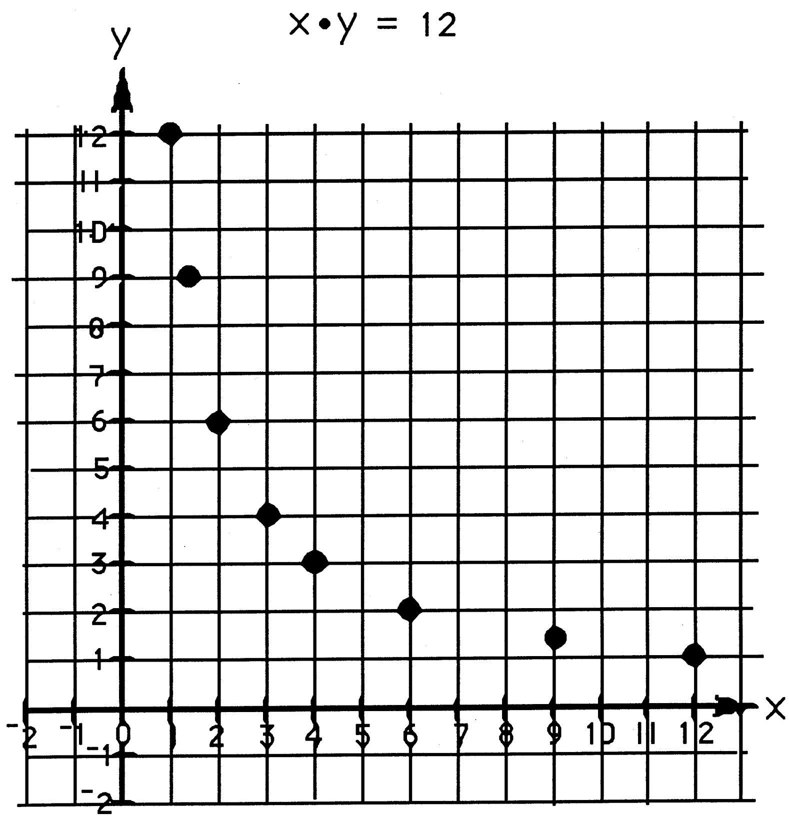
1.1 These graphs are parallel.



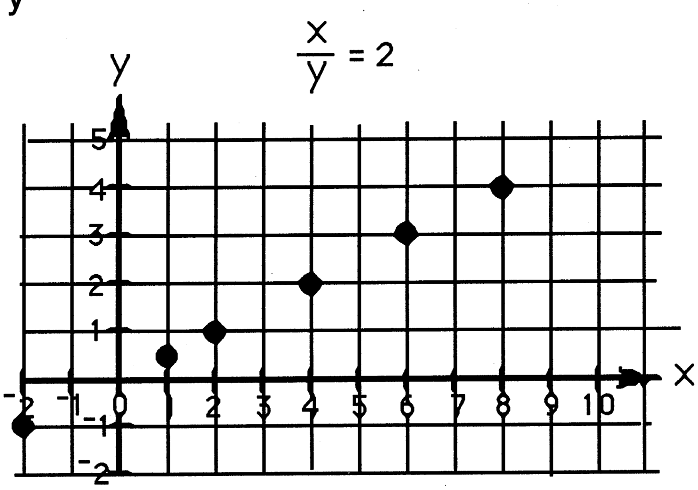
1.2 The graph of x - y = 2 is shown below:



1.3 The graph of $x * y = 12$ is shown below; this is really only half of the graph, because we haven't shown the negatives, $^-2 * {^-6} = 12$, so the point ($^-2$, $^-6$) also works. Try some negatives.



1.4 The graph of $\frac{x}{y} = 2$ is shown below:



What about the point (0,0), is that a point on the graph?

Ch. 2

2.1

6x4 + 2x8 + 9x6 = 94 goes here

$$\begin{bmatrix} 6 & 2 & 9 \\ 3 & 7 & 5 \end{bmatrix} \times \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 94 \\ 98 \end{bmatrix}$$

3x4 + 7x8 + 5x6 = 98 goes here

Notice: if we multiply a 2x3 (2-row by 3-column) matrix by a 3x1 matrix we get a 2x1 matrix for the answer.

2.2

$$\begin{bmatrix} 6 & 8 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 5 & 9 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 72 & 148 \end{bmatrix}$$

6x1 + 8x9 + 7x10 = 148

6x3 + 8x5 + 7x2 = 72

2.3

$$\begin{bmatrix} 6 & 8 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 5 & 9 \end{bmatrix} =$$

This one can't be done because there are 3 numbers in a row in the first matrix, only 2 in the columns in the second matrix!

2.4

4x10 + 6x3 = 58

4x2 + 6x8 = 56

$$\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & 10 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 56 & 58 \\ 86 & 97 \end{bmatrix}$$

7x2 + 9x8 = 86

7x10 + 9x3 = 97

2.5 The generalized case of a 2x2 by a 2x2 is:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} \underline{AE + BG} & \underline{AF + BH} \\ \underline{CE + DG} & \underline{CF + DH} \end{bmatrix}$$

2.6 Does the order of the matrices in the multiplication matter (is multiplication of matrices commutative)? Reversing the matrices above, we get

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} \times \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \underline{EA + FC} & \underline{EB + FD} \\ \underline{GA + HC} & \underline{GB + HD} \end{bmatrix}$$

Looking at the two answers, the number in the first row and first column of the answer matrices, $AE + BG \neq EA + FC$, so the multiplication of matrices is not commutative!

2.7

$$\begin{bmatrix} 3 & 7 \end{bmatrix} \times \begin{bmatrix} 2 & 6 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 62 & 46 \end{bmatrix}$$

Ch. 3:

3.1 What matrix would change Valorie's new shape back to the original?

I guessed this matrix would change Valorie's new shape back to the original shape

Valorie's transformation matrix

$$\begin{bmatrix} X & Y \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} X & Y \end{bmatrix}$$

any point on the dog

Since
$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is the "1" matrix, which doesn't change the original.

and
$$\begin{bmatrix} X & Y \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix}$$

So my guess matrix was correct.

Another way to show if my guess matrix works is this:

Valorie's matrix

$$\begin{bmatrix} X & Y \end{bmatrix} \times \begin{bmatrix} 1 & {}^{-}1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} X & {}^{-}X + Y \end{bmatrix}$$

$$\begin{bmatrix} X & {}^{-}X + Y \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix}$$

and we're back to the start!

my guess matrix

Ch. 5

5.1 The matrix that rotates the shape 90° clockwise is: $\begin{bmatrix} 0 & {}^{-}1 \\ 1 & 0 \end{bmatrix}$

5.2 The matrix that rotates the shape 60° ccw is $\begin{bmatrix} \text{Cos } 60° & \text{Sin } 60° \\ {}^{-}\text{Sin } 60° & \text{Cos } 60° \end{bmatrix}$

The generalized rotation matrix is shown below:

$$\begin{bmatrix} \cos ø & \sin ø \\ {}^{-}\sin ø & \cos ø \end{bmatrix}$$

When ø =0 we get the " **1**" matrix. When ø =90° we get the " **i** " matrix. When ø =180° we get the " **⁻1**" matrix. When ø = ⁻90° (cw 90°) we get the " **⁻i** " matrix.

5.3 What happens if you multiply $\begin{bmatrix} 0 & 1 \\ {}^{-}1 & 0 \end{bmatrix}$ by itself ?

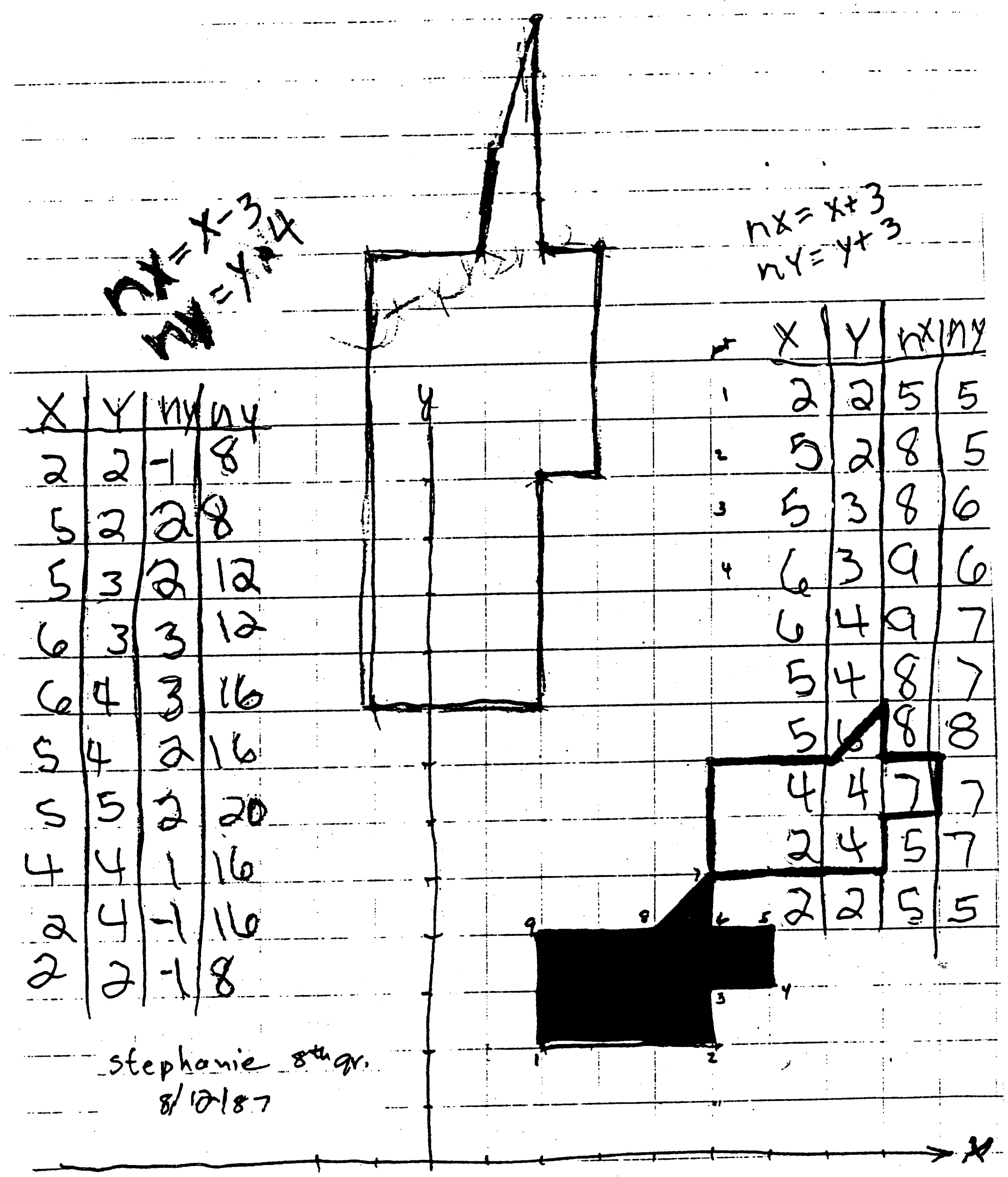
5.4 When the sides of the shape are doubled, the area is multiplied by 4. The area is proportional to the square of the length. When the sides of the shape are doubled, the perimeter is doubled. The perimeter of a new shape is proportional to the length.

How would you stretch the dog to the left, but not up?

Appendix 2: Transformations without matrices

In this section we'll make up a rule to change the dog, without using matrices. Stephanie made up **rule #1**: Add 3 to the x-number and 3 to the y-number. In other words, New X = Old X + 3 and New Y = Old Y + 3. Point #1 on the dog at (2,2) goes to (2+3,2+3) and we plot the new point at (5,5). Point #2 on the dog at (5,2) goes to (5+3,2+3) and we plot the new point at (8,5) and so on. Stephanie's **rule #1** translates the dog 3 units to the right and 3 units up, to get the new dog. This is what Andreana was trying to do.

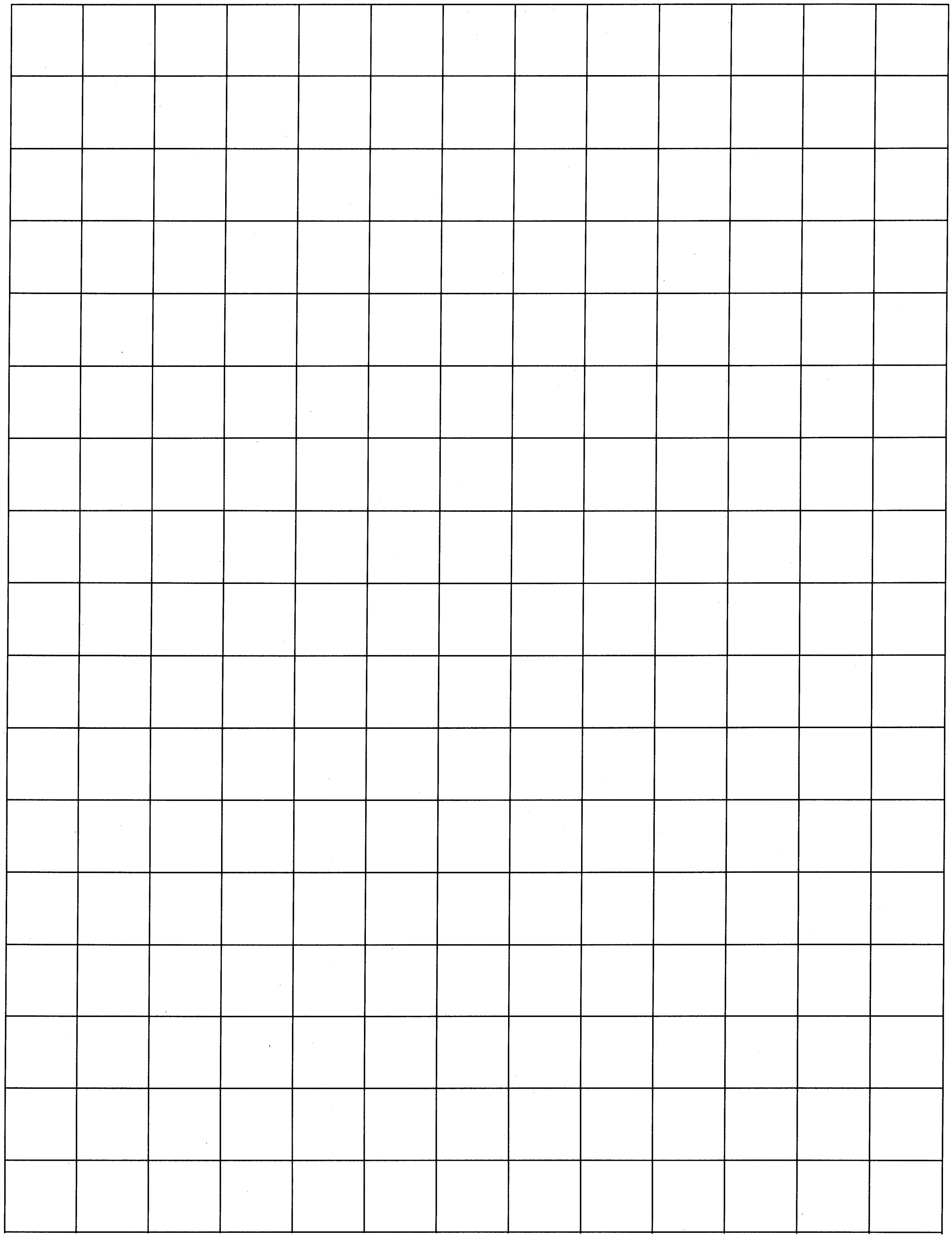
Stephanie's **rule #2** is: NewX = OldX-3 and NewY = OldY* 4. This rule moves the dog 3 units to the left and stretches its height 4 times.



X	Y	nx	ny
2	2	-1	8
5	2	2	8
5	3	2	12
6	3	3	12
6	4	3	16
5	4	2	16
5	5	2	20
4	4	1	16
2	4	-1	16
2	2	-1	8

#	X	Y	nX	nY
1	2	2	5	5
2	5	2	8	5
3	5	3	8	6
4	6	3	9	6
	6	4	9	7
	5	4	8	7
	5	5	8	8
	4	4	7	7
	2	4	5	7
	2	2	5	5

Appendix 3: Graph paper to copy for graphs and transformations



Appendix 4: Computer programs to do the transformations

Don's BASIC program to do the transformations (adapted from McGregor and Watt):

```
CLS
WINDOW 1
DEF FNsinh(x)=(EXP(x)-EXP(-x))/2
DEF FNcosh(x)=(EXP(x)+EXP(-x))/2
LET Pi=3.14159
READ noofpoints 'reads the number of points, which is the first number in the DATA list
FOR i=2 TO noofpoints
  READ x,y
   PSET (250+4*x,139-4*y) 'plots original shape
NEXT i
RESTORE 280
READ noofpoints
FOR i=2 TO noofpoints
  READ x,y
  GOSUB 240 'transform
  PSET (xt,yt)
   PSET (xt+1,yt+1)
NEXT i
END
240 'subroutine for transformation
241 LET A=2 'changes amount of "spread" of fish
242 LET m=FNsinh(x)*COS(A*y): n=FNcosh(x)*SIN(A*y) 'for non-linear
'242 LET a=10*COS(30*Pi/180): b=-10*SIN(30*Pi/180):
        c=10*SIN(30*Pi/180):d=10*COS(30*Pi/180) 'rotation of 30°
'242 LET a=10:b=0:c=0:d=10 'in this case for scaling
250    xt=250 + 3*m 'for non-linear transformation
260    yt=139-3*n 'for non-linear transformation
'250    xt=250+a*x+b*y 'for linear transformation
'260    yt=139-(c*x+d*y) 'for linear transformation
270 RETURN

'280 DATA 11,0,0,0,0,3,0,3,1,4,1,4,2,3,2,3,3,2,2,0,2,0,0,0 'dog
280 DATA 16,0,0,1,0.5,2,1,2.5,1,3,1,3.5,0.5,4,0.5,4,0,4,-0.5,3.5,-0.5,3,-1,2.5,-1,2,-1,1,
              -0.5,0,0,0,0 'fish
'280 DATA 11,0,0,0,-2,1,-2,2,-2,3,-2,3,2,2,2,1,2,0,2,0,1,0,0,0,0 'rectangle
```

Theo's *Mathematica* ® program to do the non-linear transformation (like the fish):

```
HyperEllipticTransform[{x_, y_}, a_] :={
  Sinh[x Degree] Cos[a y Degree],
  Cosh[x Degree] Sin[a y Degree]};

IdentityTransform[{x_, y_}, a_]:={x, y};

theTransform = HyperEllipticTransform;

MakeGrid[points_] := {Map[Line, points], Map[Line, Transpose[points]]};

gridPoints =
  Table[theTransform[{x, y}, 2.5], {x, -4, 44, 2}, {y, -18, 18, 2}];

fishPoints = {{0, 0}, {30, 0}, {30, 10}, {40, 10},
    {40, 20}, {30, 20}, {30, 30}, {20, 20}, {0, 20}, {0, 0}};

subdivide[{{x1_, y1_}, {x2_, y2_}}] :=
  Table[{x1 + i (x2 - x1), y1 + i (y2 - y1)}, {i, 0, 1, 0.1}]

fishPoints1 = Flatten[Map[subdivide, Partition[fishPoints, 2, 1]], 1]

{{0, 0}, {3., 0}, {6., 0}, {9., 0}, {12., 0}, {15., 0}, {18., 0}, {21., 0},
  {24., 0}, {27., 0}, {30., 0}, {30, 0}, {30, 1.}, {30, 2.}, {30, 3.}, {30, 4.},
  {30, 5.}, {30, 6.}, {30, 7.}, {30, 8.}, {30, 9.}, {30, 10.}, {30, 10},
  {31., 10}, {32., 10}, {33., 10}, {34., 10}, {35., 10}, {36., 10}, {37., 10},
  {38., 10}, {39., 10}, {40., 10}, {40, 10}, {40, 11.}, {40, 12.}, {40, 13.},
  {40, 14.}, {40, 15.}, {40, 16.}, {40, 17.}, {40, 18.}, {40, 19.}, {40, 20.},
  {40, 20}, {39., 20}, {38., 20}, {37., 20}, {36., 20}, {35., 20},
  {34., 20}, {33., 20}, {32., 20}, {31., 20}, {30., 20}, {30, 20},
  {30, 21.}, {30, 22.}, {30, 23.}, {30, 24.}, {30, 25.}, {30, 26.},
  {30, 27.}, {30, 28.}, {30, 29.}, {30, 30.}, {30, 30}, {29., 29.},
  {28., 28.}, {27., 27.}, {26., 26.}, {25., 25.}, {24., 24.}, {23., 23.},
  {22., 22.}, {21., 21.}, {20., 20.}, {20, 20}, {18., 20}, {16., 20},
  {14., 20}, {12., 20}, {10., 20}, {8., 20}, {6., 20}, {4., 20}, {2., 20},
  {-1.73472 10^-18, 20}, {0, 20}, {0, 18.}, {0, 16.}, {0, 14.}, {0, 12.},
  {0, 10.}, {0, 8.}, {0, 6.}, {0, 4.}, {0, 2.}, {0, -1.73472 10^-18}}

Show[Graphics[{MakeGrid[gridPoints], Thickness[0.02],
    Line[Map[theTransform[# - {0, 15}, 2.5]&, fishPoints1]]}],
  AspectRatio -> Automatic];
```

Theo's *Mathematica* ® program to do the 81-2x2 matrices and its transformation:

■ Draw Transformed Points with *Mathematica*

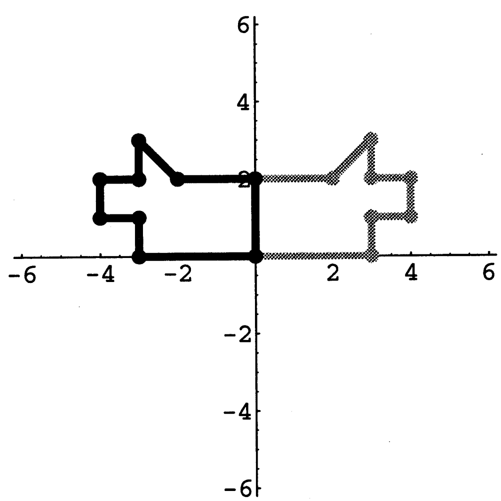
```
dog:={{0, 0}, {3, 0}, {3, 1}, {4, 1}, {4, 2},
{3, 2}, {3, 3}, {2, 2}, {0, 2}, {0, 0}};

DrawTransformedPoints[points_, matrix_] :=
      Show[Graphics[
            {
                  Thickness[0.015],
                  PointSize[0.03],

                  GrayLevel[0.75],
                  Map[Point, points],
                  Line[points],

                  GrayLevel[0],
                  Map[Point[(# . matrix)]&, points],
                  Line[Map[(# . matrix)&, points]]
            },
            PlotRange -> {{-6.2, 6.2}, {-6.2, 6.2}},
            Axes->Automatic,
            AspectRatio -> Automatic
      ]];

DrawTransformedPoints[dog, {{-1, 0}, {0, 1}}];
```



Appendix 5: Bibliography

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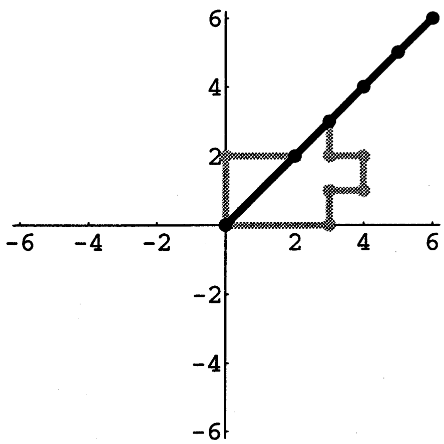
Appendix 6: The 81- 2x2 matrices using only 1's, 0's or $^{-}1$'s, the transformations and their rules.

After asking my students to make up a 2x2 matrix with 1's, 0's and $^{-}1$'s only, I started making each one, with its transformation of the dog. I used small pieces of graph paper and did about 20 of them when I asked the question, how many matrices will there be? I started to put them in order. Then I realized I had 3 numbers to put into 4 spaces, and this should be $3^4 = 81$ matrices, which Andreana figured out also. I then realized I could find $(a+b+c)^4$ in *Derive* and get some order:

$$(a+b+c)^4 = 1*a^4 + 4*a^3*b + 4*a^3*c + 6*a^2*b^2 + 12*a^2*b*c + 6*a^2*c^2 + 4*a*b^3 + 12*a*b^2*c + 12*a*b*c^2 + 4*a*c^3 + 1*b^4 + 4*b^3*c + 6*b^2*c^2 + 4*b*c^3 + 1*c^4$$

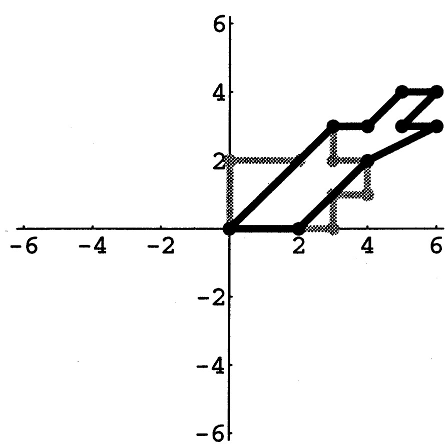
adding up the coefficients we get 81. If a= 1, b=0 and c=$^{-}1$, then this reads 1 matrix with four 1's, 4 matrices with three 1's and one 0, 4 matrices with three 1's and one $^{-}1$, and so on. At this point I realized that my diagrams wouldn't be good enough to put in a book and too much work. I called Theo who then did these in *Mathematica* for me. The program he used is in appendix 4, the 81 matrices, transformations and their rules follow. Theo thought of the matrices as having 4 counters, which gave a different order than the one I had. Each of the following 27 pages has the 'master' matrix up above to the right or left depending on whether it is on an odd or even page. The 'master' matrix is in the form $\begin{bmatrix} 1 & 1 \\ 1 & X \end{bmatrix}$, where x changes from 1 to 0 to $^{-}1$ for matrices #s1, 2 and 3, for example. Then the number in the lower left corner changes to 0 then $^{-}1$, giving matrices #'s4-9 .Then the number in the upper right goes back to 1 and the number in the lower left changes to 0, then $^{-}1$ to give matrices #s10-18, and so on. Theo thought of the 2x2 matrix as having 4 counters which changed.

$$\begin{bmatrix} 1 & 1 \\ 1 & x \end{bmatrix}$$



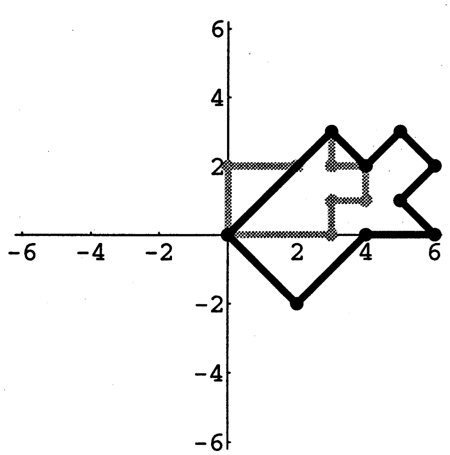
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ #1

newX = oldX + oldY
newY = oldX + oldY



$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$ #2

newX = oldX + oldY
newY = oldX



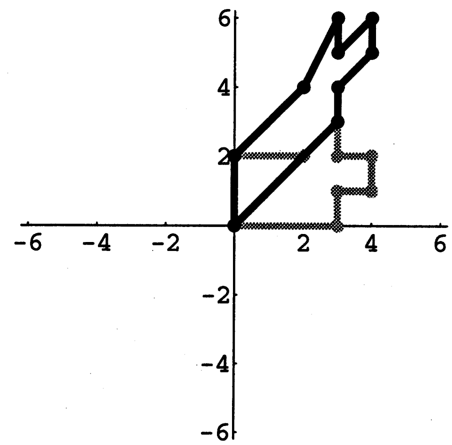
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ #3

newX = oldX + oldY
newY = oldX - oldY

$$\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}$$

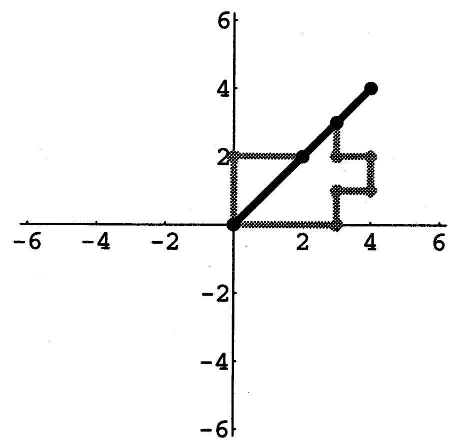
#4 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

newX = oldX
newY = oldX + oldY



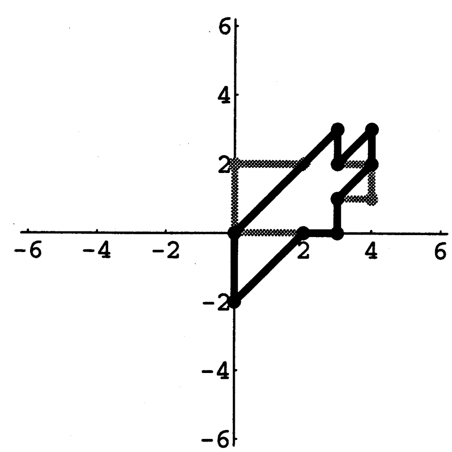
#5 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

newX = oldX
newY = oldX

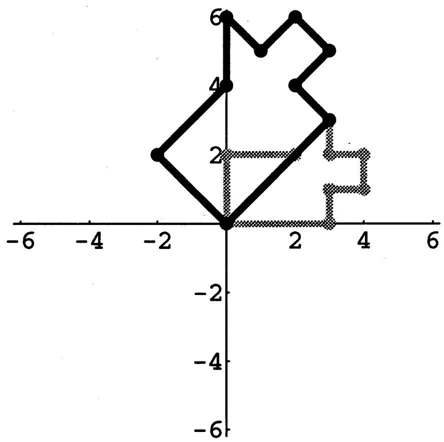


#6 $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

newX = oldX
newY = oldX - oldY

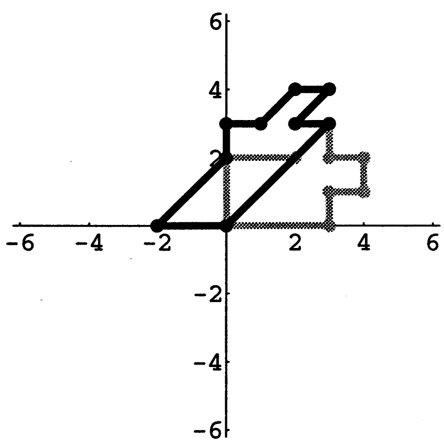


$$\begin{bmatrix} 1 & 1 \\ -1 & x \end{bmatrix}$$



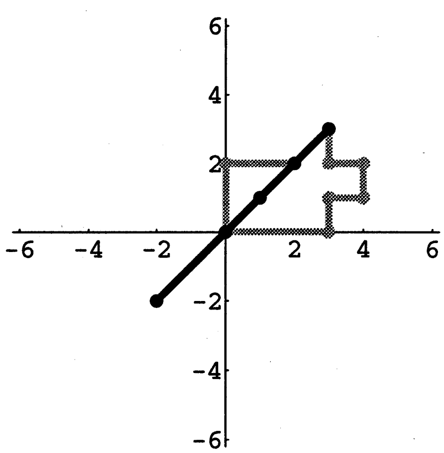
$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$ #7

newX = oldX - oldY
newY = oldX + oldY



$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$ #8

newX = oldX - oldY
newY = oldX



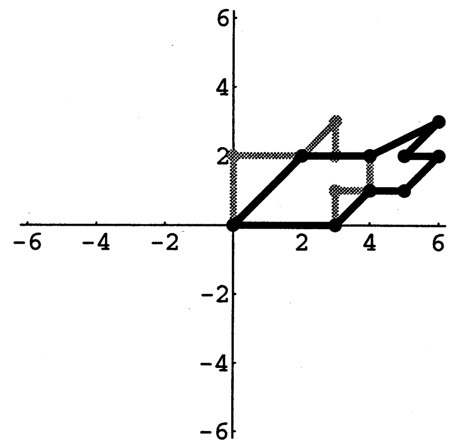
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$ #9

newX = oldX - oldY
newY = oldX - oldY

$$\begin{bmatrix} 1 & 0 \\ 1 & x \end{bmatrix}$$

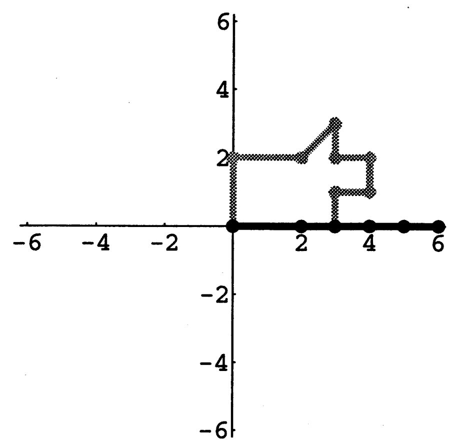
#10 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

newX = oldX + oldY
newY = oldY



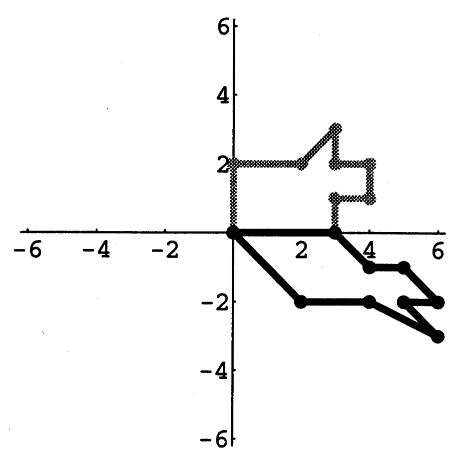
#11 $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

newX = oldX + oldY
newY = 0

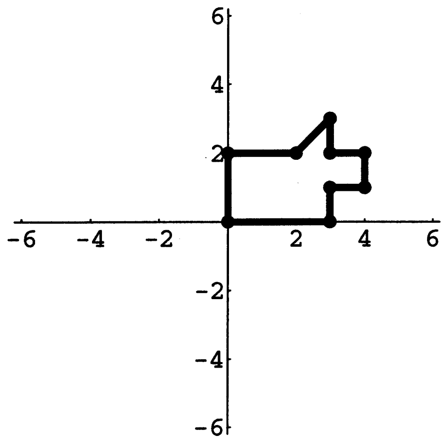


#12 $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

newX = oldX + oldY
newY = -oldY

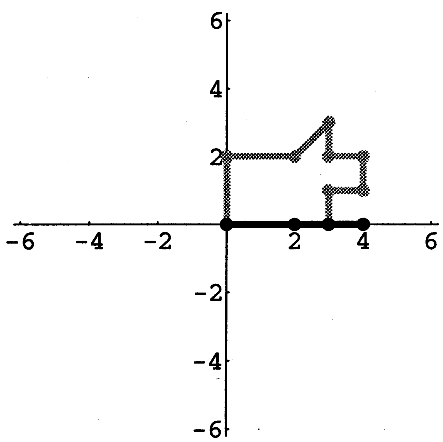


$$\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$$



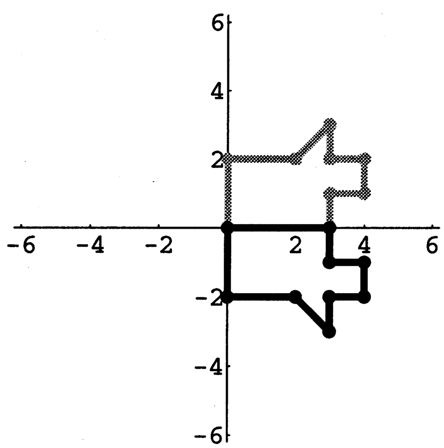
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ #13

newX = oldX
newY = oldY



$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ #14

newX = oldX
newY = 0



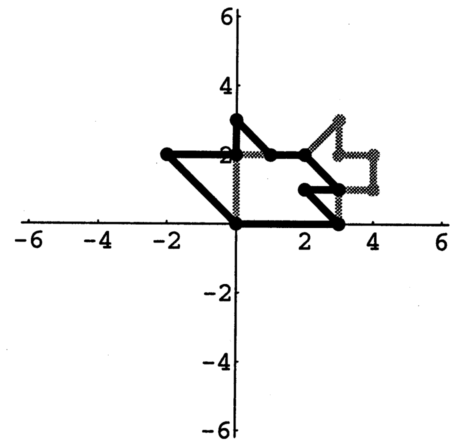
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ #15

newX = oldX
newY = -oldY

$$\begin{bmatrix} 1 & 0 \\ -1 & x \end{bmatrix}$$

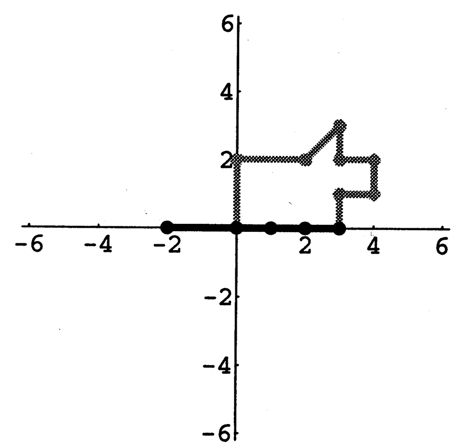
#16 $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

newX = oldX - oldY
newY = oldY



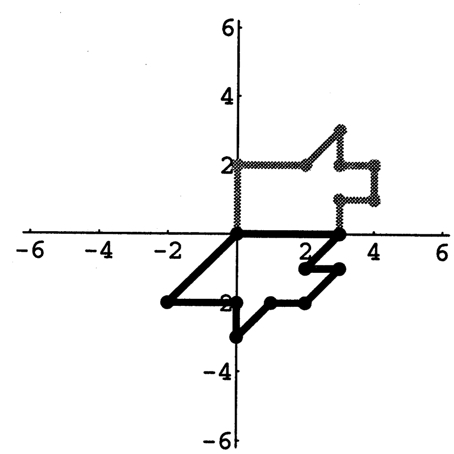
#17 $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

newX = oldX - oldY
newY = 0

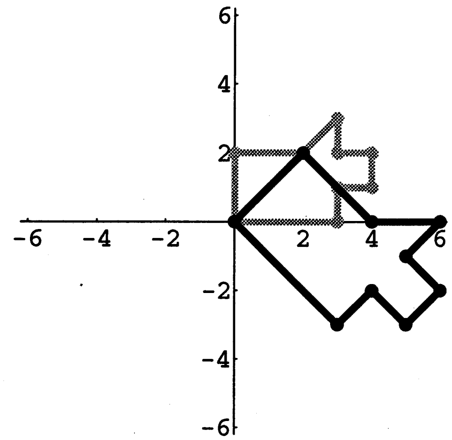


#18 $\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

newX = oldX - oldY
newY = -oldY

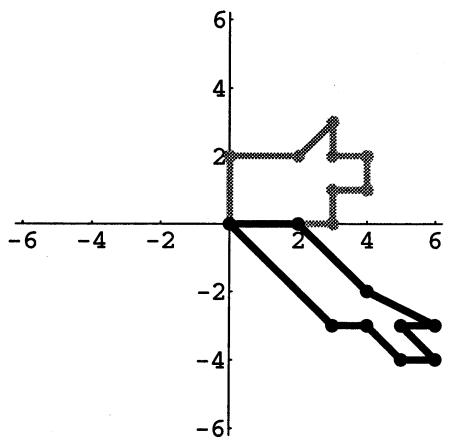


$$\begin{bmatrix} 1 & -1 \\ 1 & x \end{bmatrix}$$



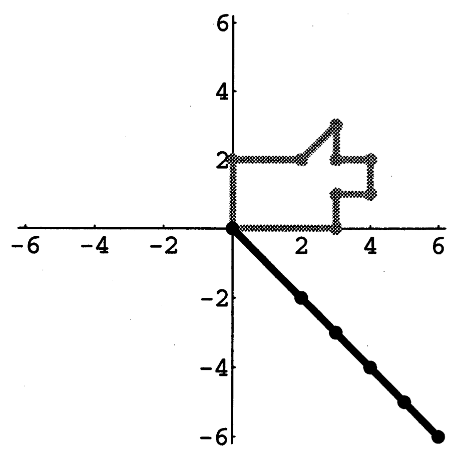
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$ #19

newX = oldX + oldY
newY = -oldX + oldY



$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$ #20

newX = oldX + oldY
newY = -oldX



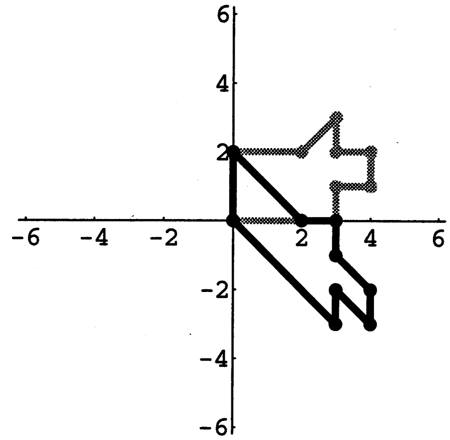
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$ #21

newX = oldX + oldY
newY = -oldX - oldY

$$\begin{bmatrix} 1 & -1 \\ 0 & x \end{bmatrix}$$

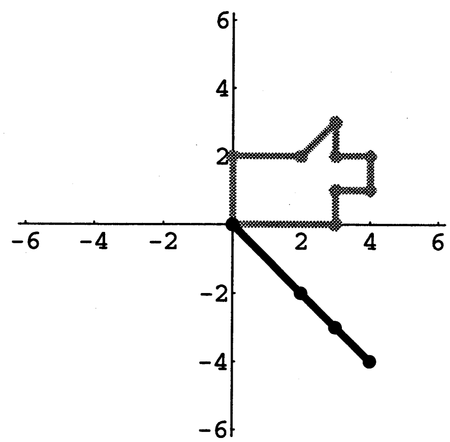
#22 $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

newX = oldX
newY = -oldX + oldY



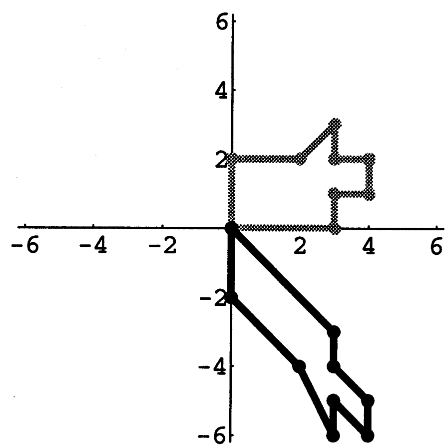
#23 $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

newX = oldX
newY = -oldX

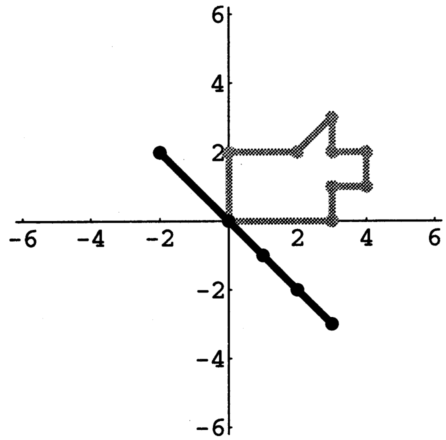


#24 $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

newX = oldX
newY = -oldX - oldY

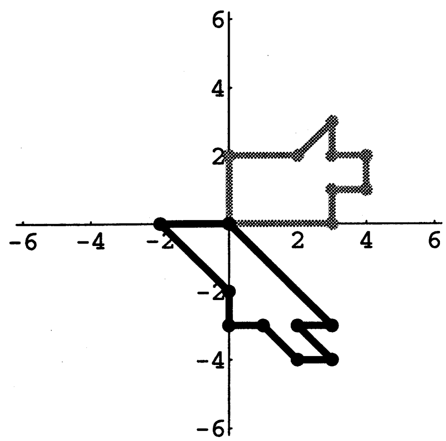


$$\begin{bmatrix} 1 & -1 \\ -1 & x \end{bmatrix}$$



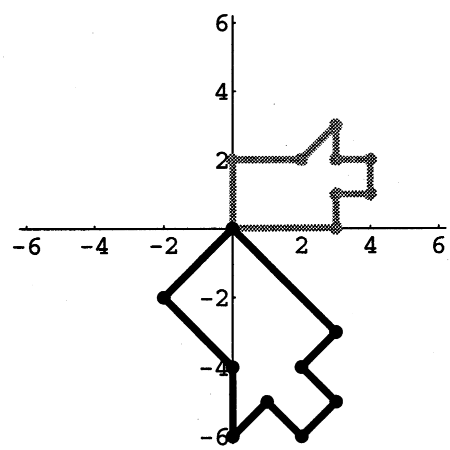
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ #25

newX = oldX - oldY
newY = -oldX + oldY



$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$ #26

newX = oldX - oldY
newY = -oldX



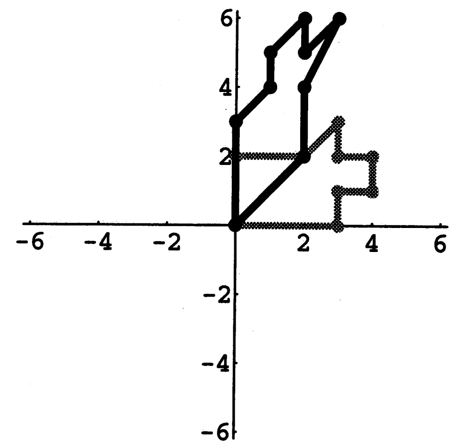
$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$ #27

newX = oldX - oldY
newY = -oldX - oldY

$$\begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix}$$

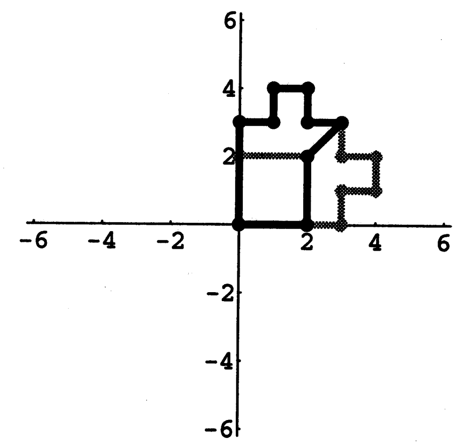
#28 $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

newX = oldY
newY = oldX + oldY



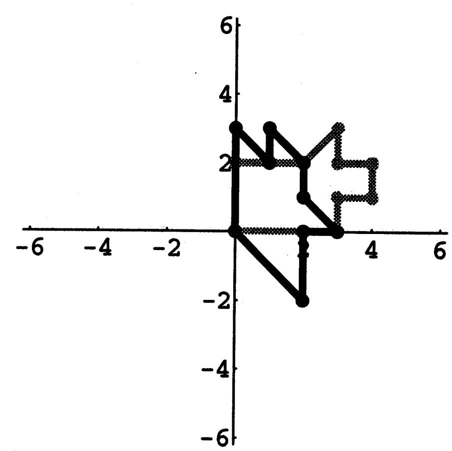
#29 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

newX = oldY
newY = oldX

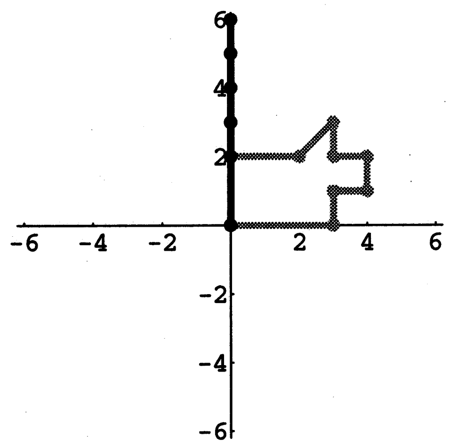


#30 $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

newX = oldY
newY = oldX - oldY

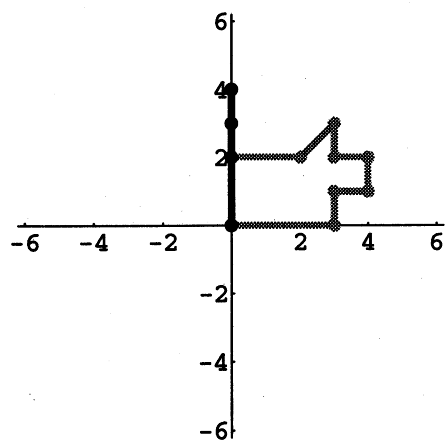


$$\begin{bmatrix} 0 & 1 \\ 0 & x \end{bmatrix}$$



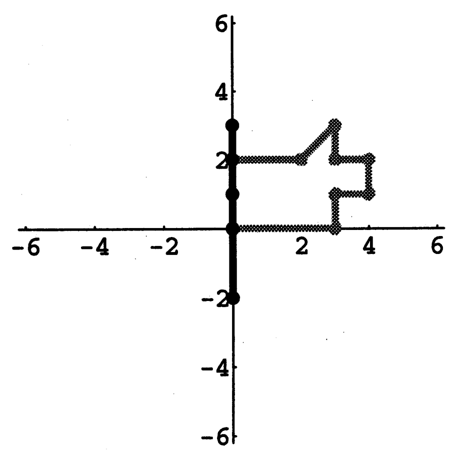
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \#31$$

newX = 0
newY = oldX + oldY



$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \#32$$

newX = 0
newY = oldX



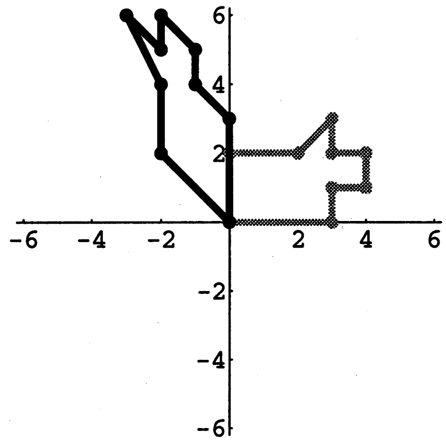
$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad \#33$$

newX = 0
newY = oldX - oldY

$$\begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix}$$

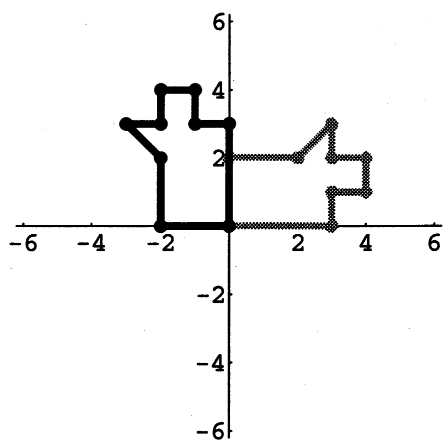
#34 $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

newX = -oldY
newY = oldX + oldY



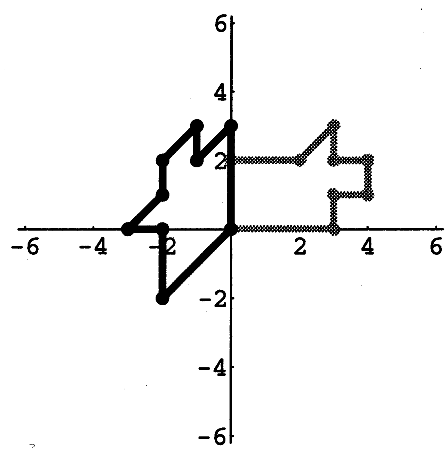
#35 $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

newX = -oldY
newY = oldX

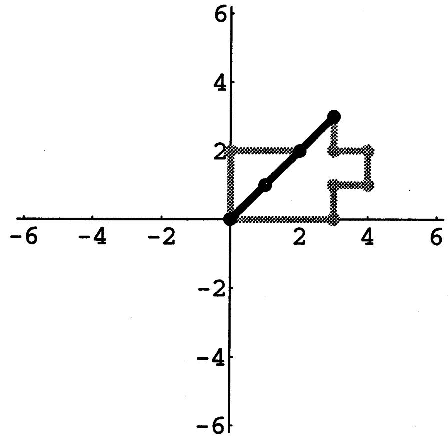


#36 $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

newX = -oldY
newY = oldX - oldY

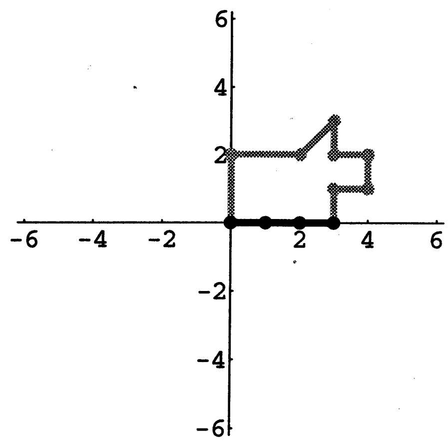


$$\begin{bmatrix} 0 & 0 \\ 1 & x \end{bmatrix}$$



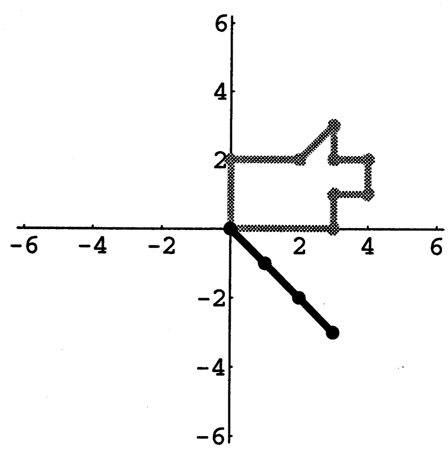
$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$ #37

newX = oldY
newY = oldY



$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$ #38

newX = oldY
newY = 0



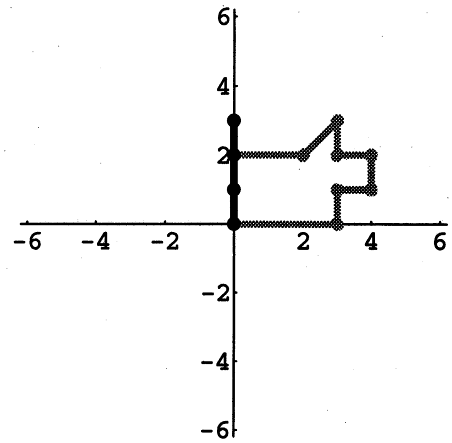
$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$ #39

newX = oldY
newY = -oldY

$$\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$$

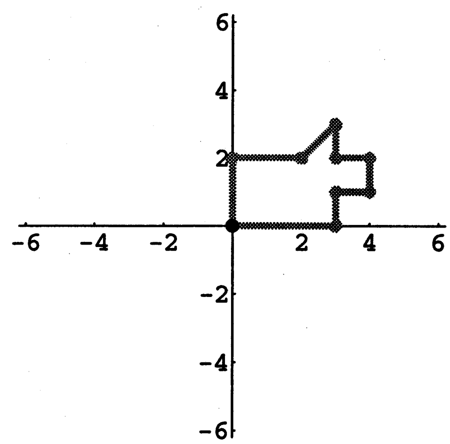
#40 $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

newX = 0
newY = oldY



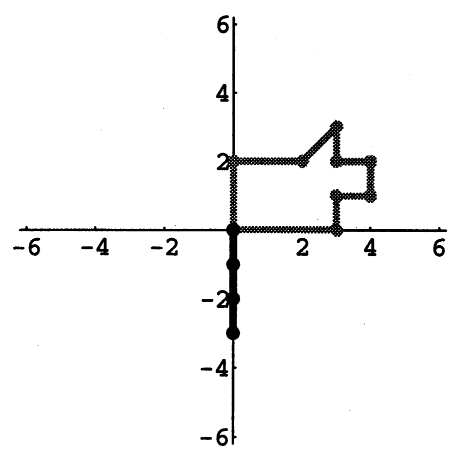
#41 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

newX = 0
newY = 0

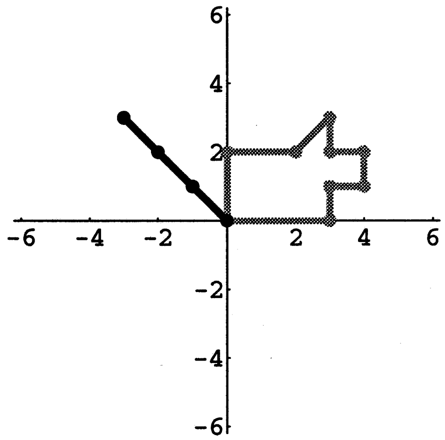


#42 $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

newX = 0
newY = -oldY



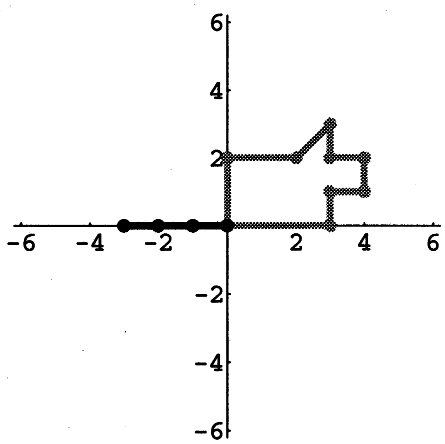
$$\begin{bmatrix} 0 & 0 \\ -1 & x \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

#43

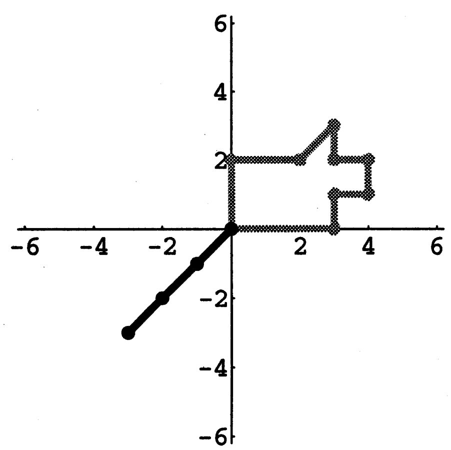
newX = -oldY
newY = oldY



$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

#44

newX = -oldY
newY = 0



$$\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$

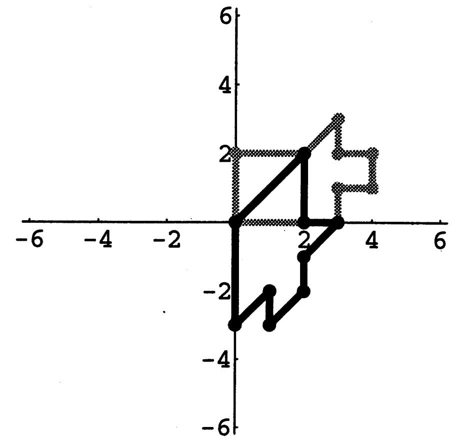
#45

newX = -oldY
newY = -oldY

$$\begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix}$$

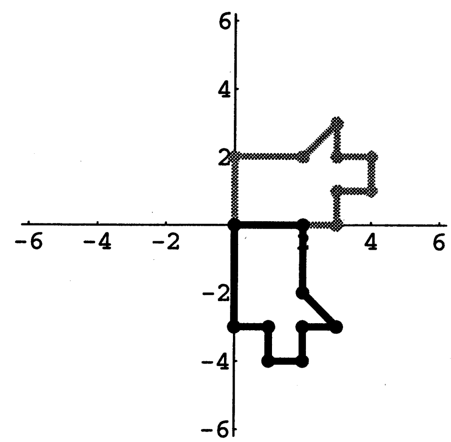
#46 $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

newX = oldY
newY = -oldX + oldY



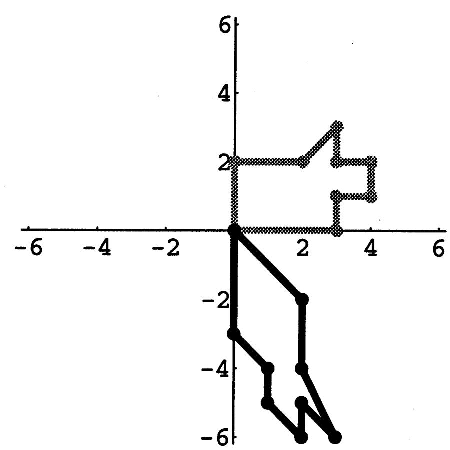
#47 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

newX = oldY
newY = -oldX

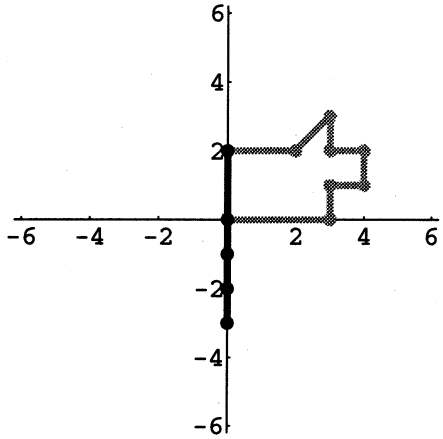


#48 $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

newX = oldY
newY = -oldX - oldY

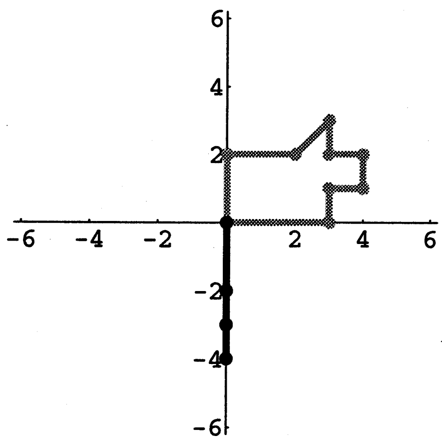


$$\begin{bmatrix} 0 & -1 \\ 0 & x \end{bmatrix}$$



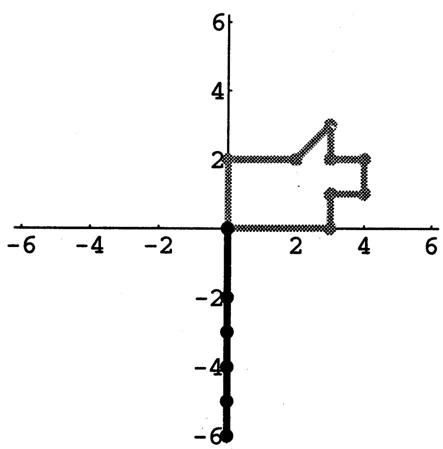
$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$ #49

newX = 0
newY = -oldX + oldY



$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$ #50

newX = 0
newY = -oldX



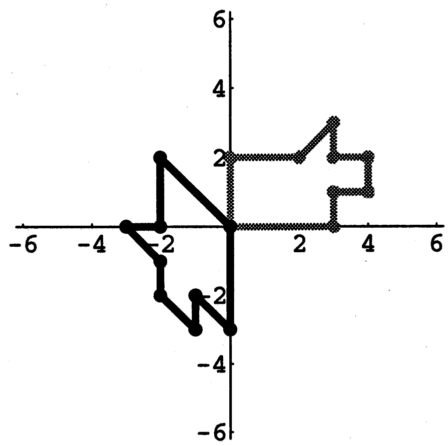
$$\begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$ #51

newX = 0
newY = -oldX - oldY

$$\begin{bmatrix} 0 & -1 \\ -1 & x \end{bmatrix}$$

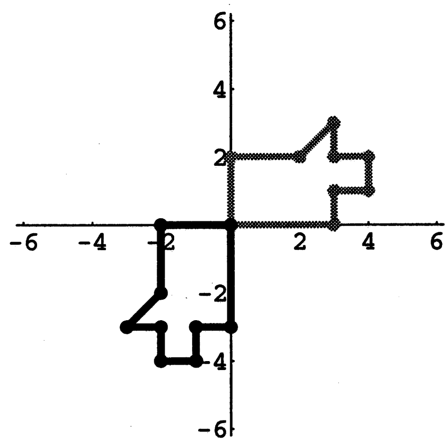
#52 $\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$

newX = -oldY
newY = -oldX + oldY



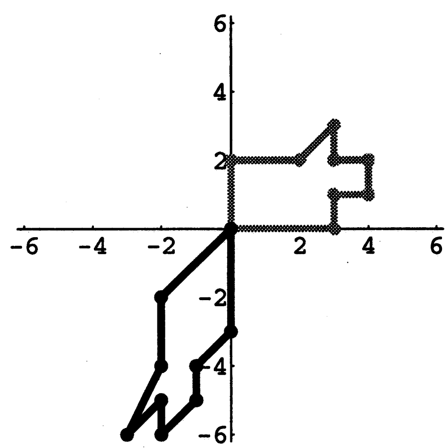
#53 $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

newX = -oldY
newY = -oldX

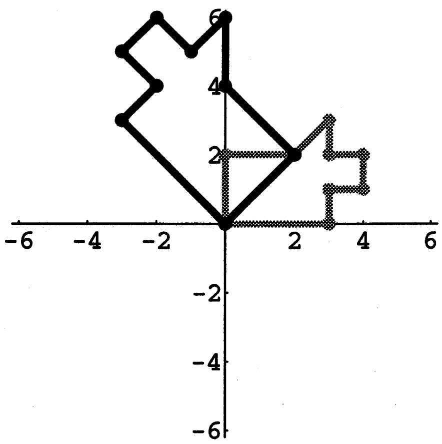


#54 $\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$

newX = -oldY
newY = -oldX - oldY

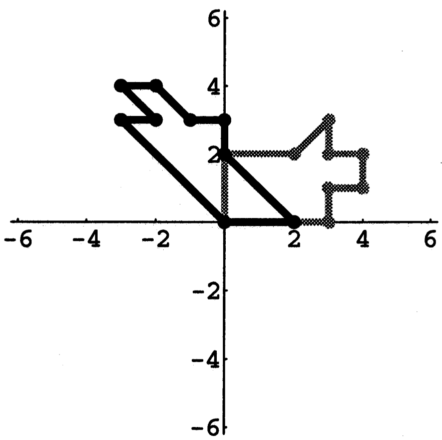


$$\begin{bmatrix} -1 & 1 \\ 1 & x \end{bmatrix}$$



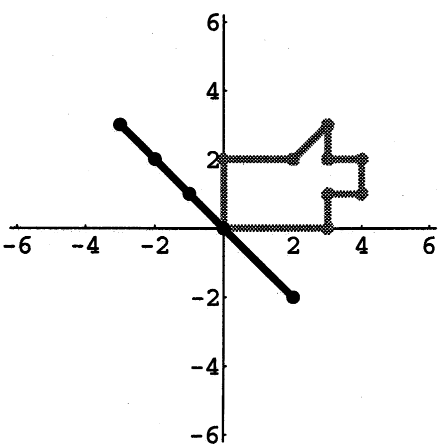
$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$ #55

newX = -oldX + oldY
newY = oldX + oldY



$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$ #56

newX = -oldX + oldY
newY = oldX



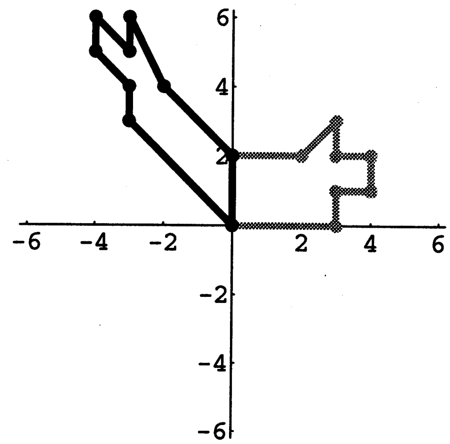
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$ #57

newX = -oldX + oldY
newY = oldX - oldY

$$\begin{bmatrix} -1 & 1 \\ 0 & x \end{bmatrix}$$

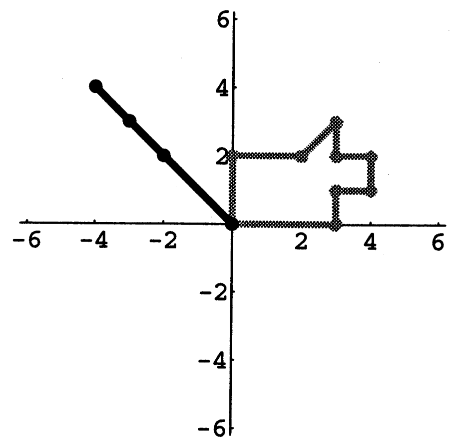
#58 $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

newX = -oldX
newY = oldX + oldY



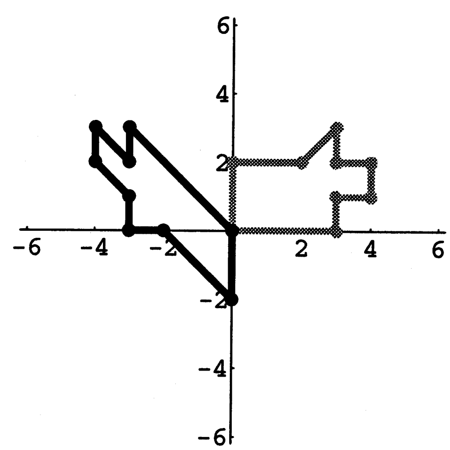
#59 $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

newX = -oldX
newY = oldX

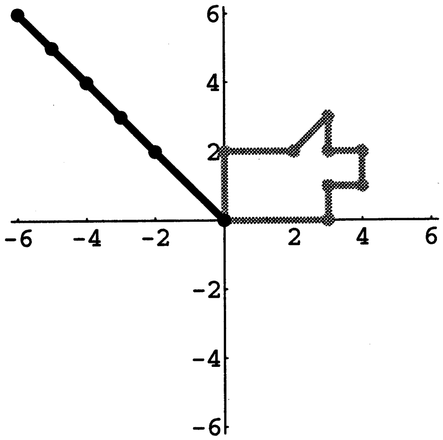


#60 $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

newX = -oldX
newY = oldX - oldY

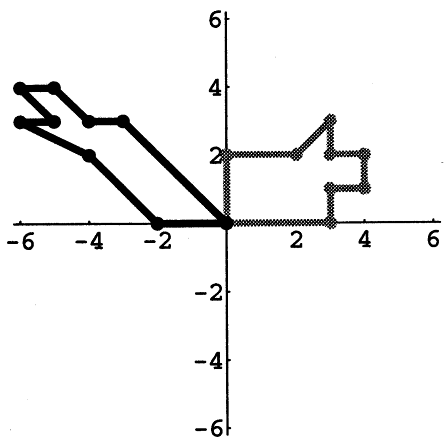


$$\begin{bmatrix} -1 & 1 \\ -1 & x \end{bmatrix}$$



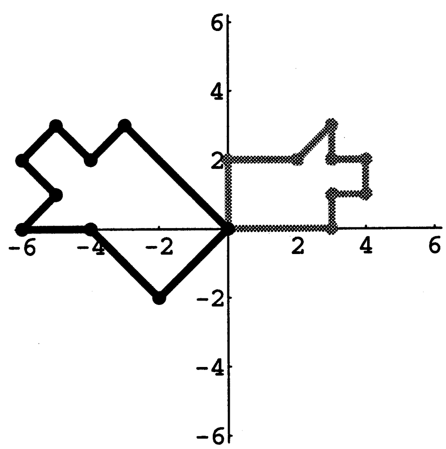
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$ #61

newX = -oldX - oldY
newY = oldX + oldY



$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$ #62

newX = -oldX - oldY
newY = oldX



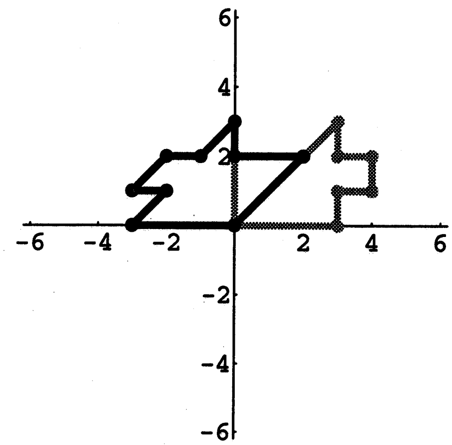
$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$ #63

newX = -oldX - oldY
newY = oldX - oldY

$$\begin{bmatrix} -1 & 0 \\ 1 & x \end{bmatrix}$$

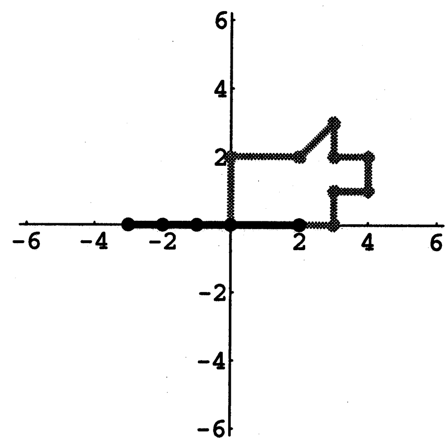
#64 $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

newX = -oldX + oldY
newY = oldY



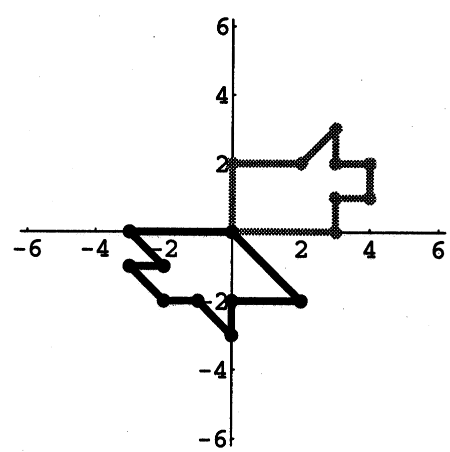
#65 $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

newX = -oldX + oldY
newY = 0

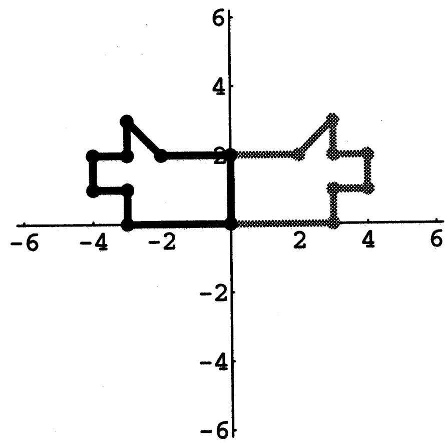


#66 $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

newX = -oldX + oldY
newY = -oldY

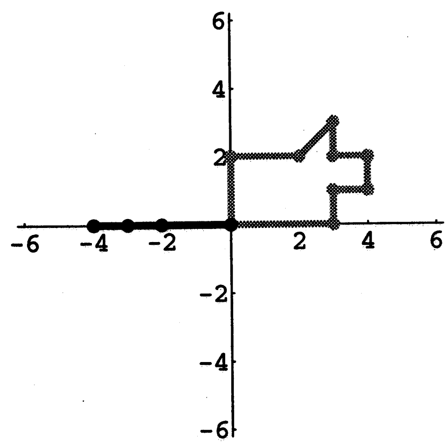


$$\begin{bmatrix} -1 & 0 \\ 0 & x \end{bmatrix}$$



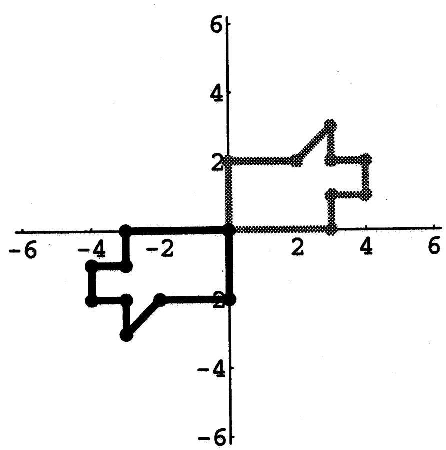
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$ #67

newX = -oldX
newY = oldY



$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$ #68

newX = -oldX
newY = 0



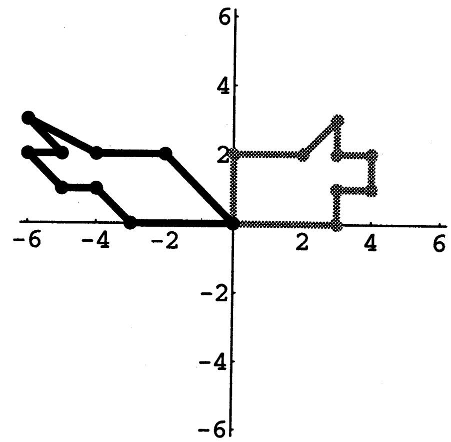
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$ #69

newX = -oldX
newY = -oldY

$$\begin{bmatrix} -1 & 0 \\ -1 & x \end{bmatrix}$$

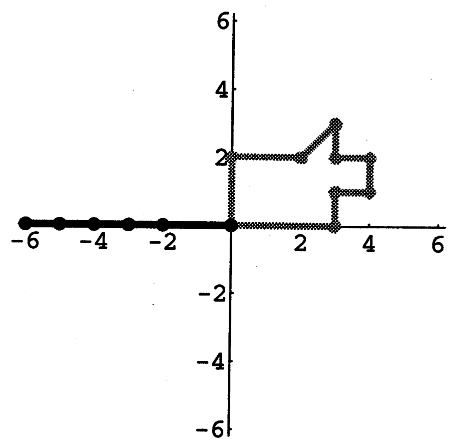
#70 $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$

newX = -oldX - oldY
newY = oldY



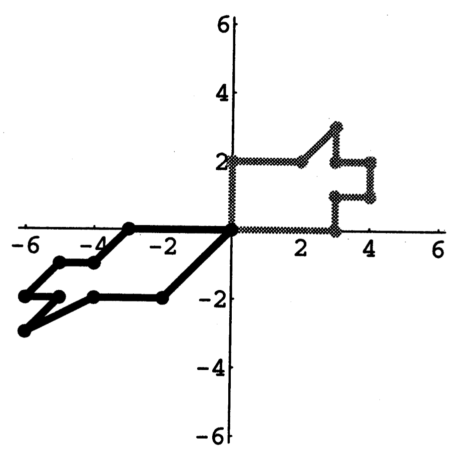
#71 $\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$

newX = -oldX - oldY
newY = 0

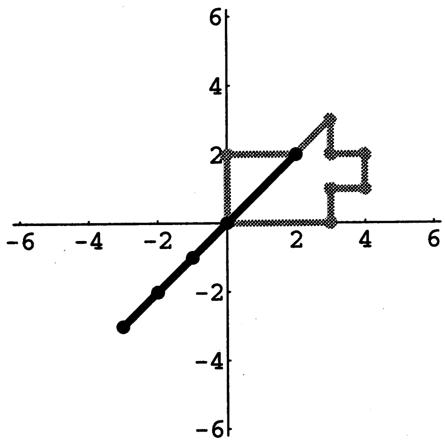


#72 $\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$

newX = -oldX - oldY
newY = -oldY

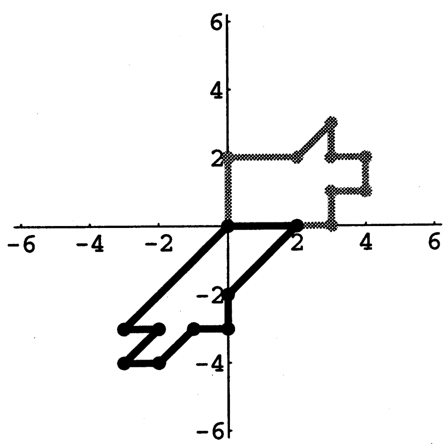


$$\begin{bmatrix} -1 & -1 \\ 1 & x \end{bmatrix}$$



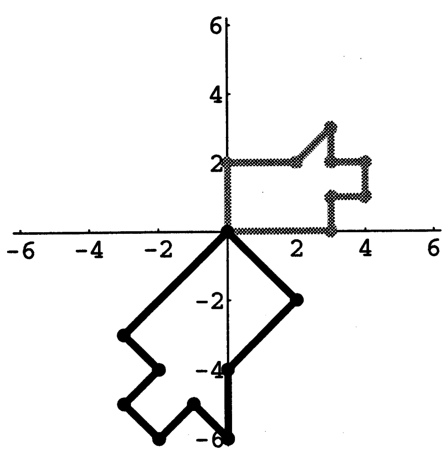
$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$ #73

newX = -oldX + oldY
newY = -oldX + oldY



$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$ #74

newX = -oldX + oldY
newY = -oldX



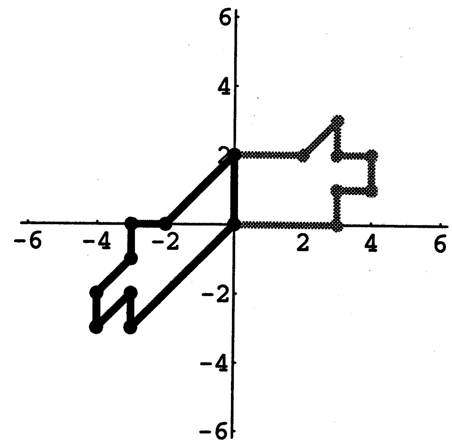
$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$ #75

newX = -oldX + oldY
newY = -oldX - oldY

$$\begin{bmatrix} -1 & -1 \\ 0 & x \end{bmatrix}$$

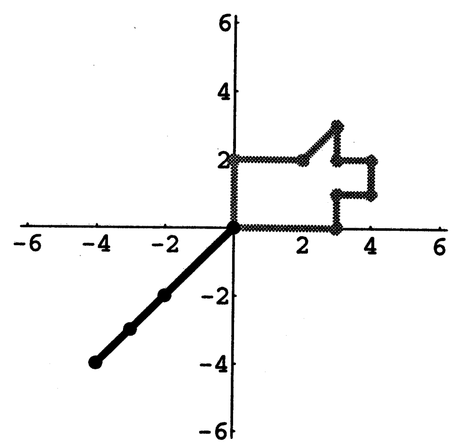
#76 $\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$

newX = -oldX
newY = -oldX + oldY



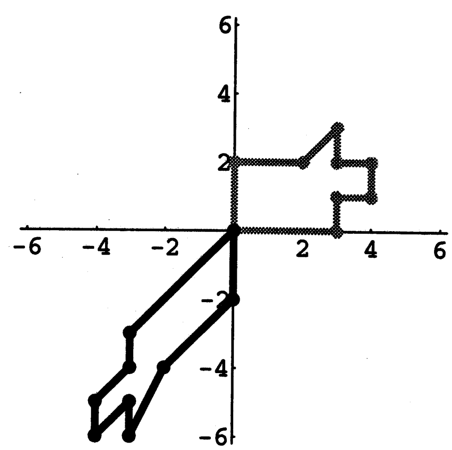
#77 $\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$

newX = -oldX
newY = -oldX

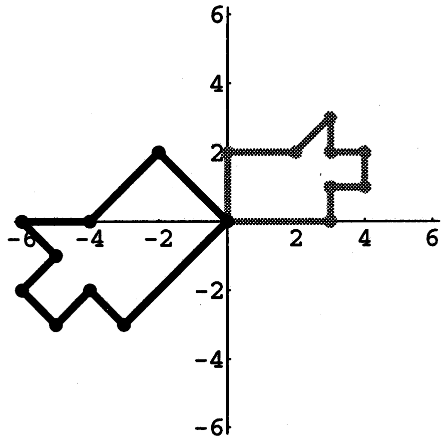


#78 $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$

newX = -oldX
newY = -oldX - oldY



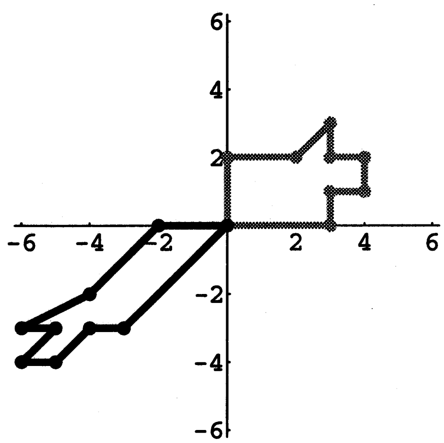
$$\begin{bmatrix} -1 & -1 \\ -1 & x \end{bmatrix}$$



$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

#79

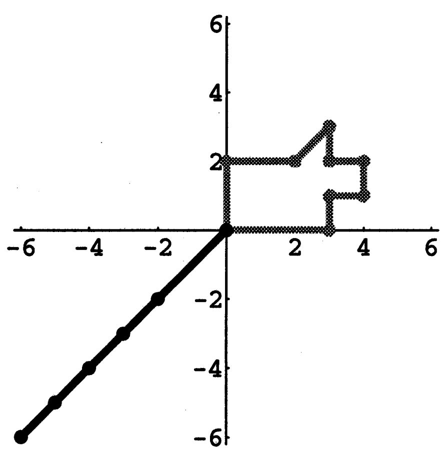
newX = -oldX - oldY
newY = -oldX + oldY



$$\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$$

#80

newX = -oldX - oldY
newY = -oldX



$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

#81

newX = -oldX - oldY
newY = -oldX - oldY

Notes

From Chapter 3

Steps to do a geometric transformations using matrices:

1. Make a shape. The shape I chose was the "dog" (simple, not too many points).

2. Pick out about 10 points on the shape.

3. Write down the coordinates of these points.

4. Number each of these points to make it easier to keep track of things.

5. Pick a 2x2 matrix using only 1's or 0's or $^-1$'s (for now). This will be your transformation

matrix. For example, Valorie chose the matrix: $\begin{bmatrix} 1 & ^-1 \\ 0 & 1 \end{bmatrix}$

6. Multiply each point (as a matrix) by your transformation matrix to get a new point.

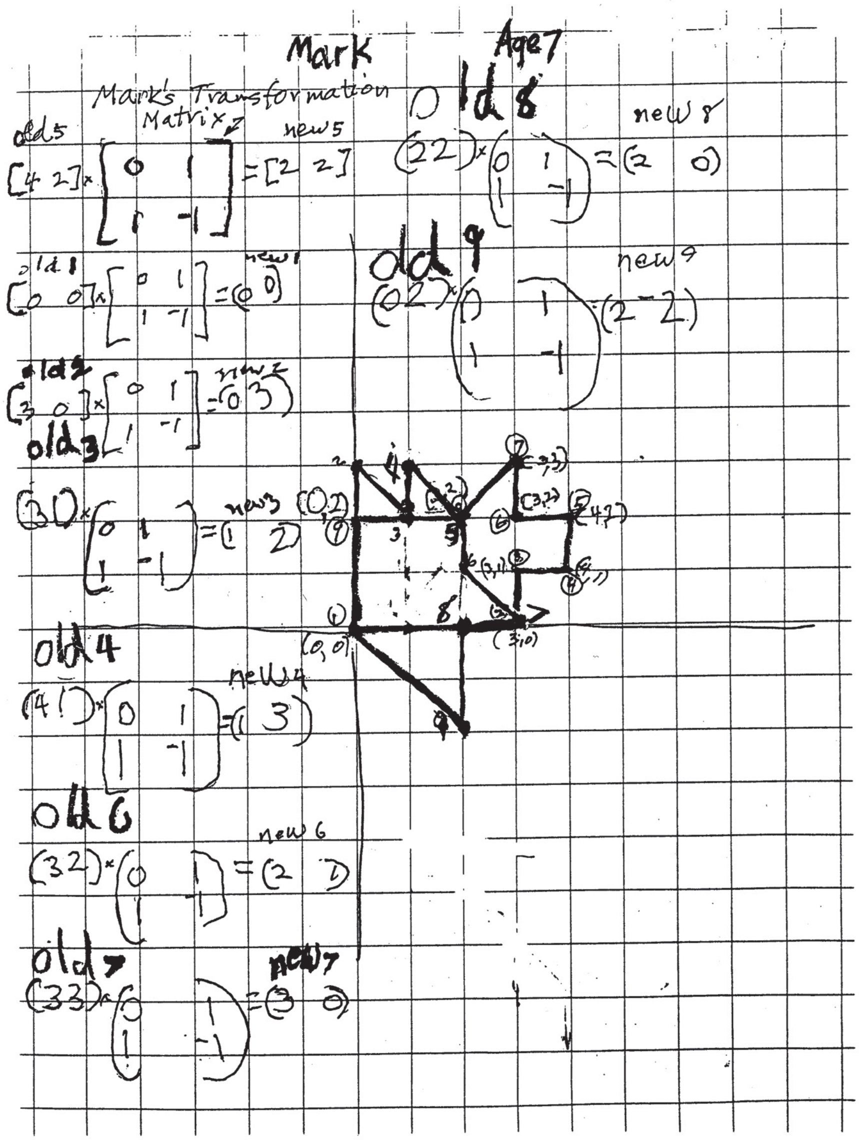
7. Plot each new point and number it, keeping a correspondence with the old points.

8. Connect the new points as you go, using a color different from the original shape.

9. Complete all the new points and close the shape (if it's a closed figure), then look at the new shape carefully. Try to figure out what your matrix did to the original shape. Then ask questions about the situation.

"Changing Shapes With Matrices"© by Don Cohen

Mark, age 7, used his transformation matrix to change the dog like this:



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