Calculus By and For Young People

(ages 7, yes 7 and up)

by Don Cohen
co-founder and teacher of
The Math Program

D Cohen 2/88

In case you missed it, the following review of this book by Phylis and Philip Morrison, appeared in the Dec. '88 issue of ***Scientific American***

"Trying to divide six cookies fairly among seven people? Third-grader Brad had the right idea: cut each one in half, share out as many as you can; again halve the pieces not shared until there are pieces enough to share, and continue. He quit at sixteenths, amidst lots of crumbs. But he could see that everyone got 1/2 + 1/4 + 0/8 + 1/16 + 1/32 + 0/64 +...of a cookie. The sum is not hard to express in terms of more familiar series, once you notice that the missing portion of unity is itself a geometric series for 1/ (1- (1/8)). Iteration is more powerful and more intuitive than dividing a round cookie into seven equal parts.

This spiral-bound book the size of your hand reports with infectious enthusiasm the work of many beginners in one fine teacher's class over the decades, some of them highly gifted kids and some of them grown-ups with no particular mathematical bent. All were on their way to an understanding of slope and integral, natural logarithm and exponential. En route a good many famous problems were encountered, among them the proof of the snaillike divergence of the harmonic series (its first million terms add up to about 14.4, a sum given here to a dozen decimals), the Fibonacci equence in pineapples and that glorious relation among, e, i, pi, 0 and 1.

(continued on the inside back cover)

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I want to thank my wife Marilyn for her constant love and support; Jerry Glynn, my partner, who helped us create a unique environment for teaching math—The Math Program, and who edited the book for me; Bob Davis who inspired me to create; all my students from whom I've learned so much, especially Ian Robertson, and all their parents who have financially enabled us to enjoy and help their children in The Math Program. My thanks also goes to Bruce Artwick on whose computer I typed this book.

I'd like to dedicate this book to a great Lady, Sadie (Nanny) LeFevre, who gave me her daughter in marriage many years ago. . . .

PREFACE

This is a book of problems for you to work on and think about. Archimedes, Newton, Euler, my students and I have worked on them also—they are important. Get out a pencil, some paper and a calculator. Expect chaos and confusion, then you'll be on your way. To supplement this book, you probably want to get Bob Davis' two books for the emphasis on algebra and graphing and Sawyer's book "What is Calculus About?". A bibliography is included later.

I was teaching a class of teachers at Webster College in the 60's; one day Judy Silver, a first-grade teacher, figured out the relationship between the derivative and the integral. I had tears come to my eyes, I was so excited. I said, why couldn't I have learned math in the way that I was teaching it now! That's why I want to share this book with young people (and their parents and teachers), so they can get an early start thinking about these ideas, and not have to memorize a lot of formulas and notations without much understanding, as I did in my mathematics courses.

"In his first paper on the Calculus (1669), Newton proudly introduced the use of infinite series to expedite the processes of the calculus . . .

As Newton, Leibnitz, the several Bernoullis, Euler, d'Alembert, Lagrange, and other 18th-century men struggled with the strange problem of infinite series and employed them in analysis, they perpetuated all sorts of blunders, made false proofs, and drew incorrect conclusions; they even gave arguments that now with hindsight we are obliged to call ludicrous."

If you have questions, discoveries to share, suggestions for changes, I would welcome hearing from you.

Start anywhere and ENJOY. But stay with it!

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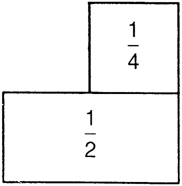
CHAPTER 1: 7 Year-Olds Do $\left(\frac{A}{B}\right) + \left(\frac{A}{B}\right)^2 + \left(\frac{A}{B}\right)^3 + \left(\frac{A}{B}\right)^4 + \ldots$

Add up the fractions below:

$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots$ or without exponents,

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$ (*forever!*)



What happens if we add these up a little at a time? You might draw a picture to show their sum.

$\frac{1}{2} = \frac{1}{2}$

$\frac{1}{2} + \frac{1}{4} =$? Since $\frac{1}{2} = \frac{2}{4}$, $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = \frac{3\frac{1}{2}}{4}$ (by Chris, a 6-year-old!)

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} =$? (What do you think Chris wrote for this one?!—he wrote the answer in $\frac{1}{4}$'s also!)

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=?$$

So the sequence of partial sums look like this:

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots$$

Can you predict the next one? Do 4 more. What patterns do you see? Is the sum getting bigger or smaller? Is $\frac{7}{8}$ bigger or smaller than $\frac{3}{4}$? Is the number we're adding each time getting bigger or smaller? Is there a number the sum is getting closer to? Will it ever get as big as 5? Is it getting closer to 5? Is it getting closer to 2? Is there a smallest number that's too big?

These sums must be increasing, because we are adding a positive number each time. The number we're adding each time is getting smaller though. As Steve said, the sums keep getting half way to one. Others have said, the sum never gets bigger than 1 because the top number is always one less than the bottom number. Someone else said, the partial sums are always 1 of the fractions less than 1. $1 = \frac{64}{64}$, and $\frac{64}{64} - \frac{1}{64} = \frac{63}{64}$. The denominators are powers of 2, $2^2 = 4$, $2^3 = 8$, etc., so one way to write the nth term of this sequence or the sum of n terms of the series is

$$1 - \left(\frac{1}{2}\right)^n$$

The sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \left(\frac{1}{2}\right)^n \ldots$ decreases and never goes below 0. So 1—(a number getting close to 0) will get closer to 1.

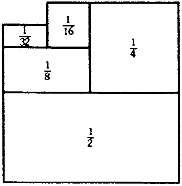
Another way to say it is, since the top number is 1 less than the bottom number

$$\frac{2^n - 1}{2^n}$$

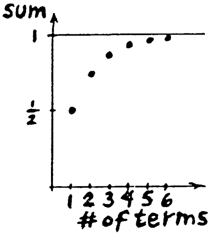
which, when we divide, we get the same answer as above. In either case, as *n* gets bigger and bigger the sum gets closer and closer to 1 and does not get bigger than 1. We'll tentatively say it equals 1.

Two ways of showing this series, geometrically, and graphically are drawn below:

Method 1: The whole thing is a square of area 1.



Method 2: A graph of the number of terms added vs. their sum.



It's interesting to look at what Chris did for a moment; he was six years old at this time. He has been with me once a week, for 45 minutes, for about 5 sessions. He drew a 16×16 square on graph paper, then cut it into pieces of $\frac{1}{2}$, $\frac{1}{4}$, etc., each piece a different color, to about $\frac{1}{4096}$. He enjoyed trying to make the pieces smaller and smaller and he had already written the sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+$$
$$\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}+\frac{1}{2048}+\frac{1}{4096}=\frac{12}{9070},$$

without any help from me. This last part was incorrect, for he was adding tops and bottoms, but he couldn't communicate why he did that. Then, not to my surprise, he wrote the name of each of these pieces as quarters and added these up. He got

$$2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+$$
$$\frac{1}{256}+\frac{1}{512}+\frac{1}{1024}=\frac{3}{2046}$$

He knew that $\frac{1}{2}=\frac{2}{4}$ and $\frac{1}{4}=\frac{1}{4}$ and $\frac{1}{8}=\frac{1}{2}$ of $\frac{1}{4}$ and $\frac{1}{16}=\frac{1}{4}$ of $\frac{1}{4}$ etc.!

That was neat. He had trouble that day trying to explain to his Mom, who had come to pick him up, and to me, why he had $\frac{3}{2046}$ as the answer and also incorrect. I had faith that he was thinking about this in a way that was very good, but he just didn't know how to write it. This is important to remember when working with children learning new things. He was thinking about $6 \div 2 = 3$ and I think it had to do with $\frac{6}{8} = \frac{3}{4}$. He also knew and wrote that

$$\frac{3\frac{1}{2}}{4} = \frac{3}{4} + \frac{\frac{1}{2}}{4} = \frac{7}{8}.$$

At one point his Mom said to me, "I don't see how this is related to what Chris is doing in school". Now the reason Chris was coming to The Math Program (a private program for which Chris' Mom and Dad pay us to work with him after school) was the fact that he was bored in school. My response was "How is Chris doing in school?" "Fine, he is still adding 2 and 3", she said. Then I explained that Chris' understanding of fractions was well beyond most 7th-graders I have worked with. We are dealing with infinite processes, which is the basis of the calculus. Besides, in doing all this, he is adding big numbers, dividing, learning about equivalent fractions and complex fractions, all of which the schools expect him to do anyway, but of course much later. I also told his Mom that we would be doing linear graphs soon (see chapter 6) and to keep asking and talking to me about her concerns. I need to understand her concerns in order to strengthen and support my belief in what I am doing as a teacher. I'm very pleased that Jerry and I have found a way to work with people so as to allow them to stretch their minds and enjoy mathematics. For those students who have difficulty we can use lots of different approaches to help them. And yesterday Chris said that all the quarters add up to $\frac{4}{4} = 1$ or close to it!

Chris does other mathematical things also. I shouldn't leave you with the impression that I have been forcing him to work only on this problem. He has made a set of Soma cubes (seven pieces of wood made from centimeter cubes and invented by Peter Hein, Parker Bros.) and has built a 3×3 cube with them. There are many other things for him to work with; he can't get bored here.

Now find this sum: $\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \ldots$

then: $\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \ldots$

and: $\left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \left(\frac{1}{5}\right)^4 + \ldots$

and the big leap: $\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \ldots$

Now you really have to look at what you have so far. Can you guess this one?

$\left(\frac{8}{17}\right) + \left(\frac{8}{17}\right)^2 + \left(\frac{8}{17}\right)^3 + \ldots$

What about $\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \ldots$?

Use a calculator at any time. Look for a pattern. Write a program on a calculator or computer to add up the terms. Can you generalize to

$\left(\frac{A}{B}\right) + \left(\frac{A}{B}\right)^2 + \left(\frac{A}{B}\right)^3 + \ldots$?

STOP HERE. TEST YOUR RESULTS. THE ANSWERS ARE ON THE NEXT PAGE.

Let's look at the sums of the infinite series we have so far:

$\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$ seems to not go above 1 and gets closer to 1, we will tentatively say it equals 1.

$$\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots = \frac{1}{2}$$
$$\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots = \frac{1}{3}$$
$$\left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \ldots = \frac{1}{4}$$
$$\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \ldots = \frac{2}{3}$$
$$\left(\frac{8}{17}\right) + \left(\frac{8}{17}\right)^2 + \left(\frac{8}{17}\right)^3 + \ldots = \frac{8}{9} = \frac{8}{17 - 8}$$

and generalizing, $\left(\frac{A}{B}\right) + \left(\frac{A}{B}\right)^2 + \left(\frac{A}{B}\right)^3 + \ldots = \frac{A}{B - A}$

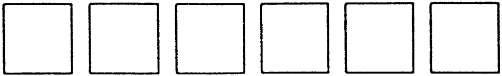
$\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \ldots$ diverges, doesn't work. These series work only if $A < B$.

Is .999999 . . . = 1? In ________, Mo., a 7th grade class, in a small building, had a great discussion about this. Students came in each day for a week with arguments that it is or is not. One day the whole class, spontaneously, made signs to put around their necks. They were going to march around the school, half the signs saying .999 . . . = 1, the other half saying it was not equal to 1. That was a most exciting and unforgettable day; marching around the school about a math problem! There was a sad ending to this story, however. The principal, upon seeing this excitement, said we couldn't do this, it would upset the school!

We can write, $.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots = 9 \cdot \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots\right)$ by factoring 9 from each term. Writing the sum with powers of ten, $9 \cdot \left[\left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \ldots\right]$. This has an infinite series in it which we know how to deal with now, and equals $9 \cdot \left[\frac{1}{10 - 1}\right] = 9 \cdot \left(\frac{1}{9}\right) = 1$. So $.999999999\ldots = 1$. Can you change these infinite repeating decimals to fractions: a) .18181818 . . . , b) .135135135 . . . and c) .037373737 . . .

CHAPTER 2: Brad's: Share 6 Cookies With 7 People.

If each person is to get an equal share, how many cookies does each person get? Use 3 $\times$ 5 cards as cookies (not circles) and use scissors to solve this problem yourself before going on.

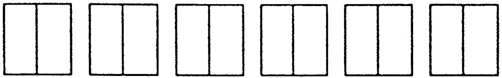


The nice thing about this is that each person can come up with a different way to solve it, and each can be correct.

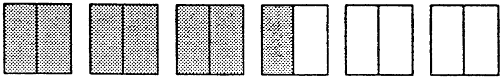
Be careful about the size pieces you have as you go along. You have to keep asking, how many of this size piece makes a whole cookie? Then you can name each piece. For example, if 16 of the pieces make a whole cookie, then each piece is $\frac{1}{16}$ of a cookie.

Brad, who had just finished 2nd grade, solved it this way: his rule was, I'll cut each cookie in half, share them if I can, if not, cut these pieces in half until I can share again, and continue to do that.

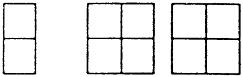
Here we go (you might want to do this with your 3×5 cards):
He couldn't share 6 cookies with 7 people, so he cut each in half, obtaining 12 half-pieces.



He shared these pieces; each person got $\frac{1}{2}$ of a cookie. He had 5 halves left, which he couldn't share with 7 people.

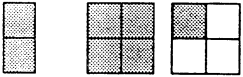


He cut these 5 pieces in half.
He now had 10 quarters

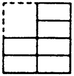


He shared these with the 7 people.

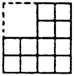
Each person now had $\frac{1}{2} + \frac{1}{4}$ of a cookie. There were 3 quarter-pieces left over, which he couldn't share with the 7 people.



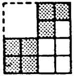
He cut these 3 quarter-pieces in half, which resulted in his having 6 eighth-pieces. But this was not enough to share with the 7 people.



Now each person still had $\frac{1}{2} + \frac{1}{4}$.
He cut the 6 eighth-pieces in half, obtaining 12 sixteenth-pieces.
Each person then shared one of these.



Now each person had $\frac{1}{2} + \frac{1}{4} + \frac{1}{16}$ of a cookie and he said he could go on (and get lots of crumbs).



There is a pattern; each person gets:

$$\frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16} + \frac{1}{32} + \frac{0}{64} + \ldots \text{ of a cookie!}$$

And we have an infinite series!

Brad did something I had never seen before. Looking at things in a new way, seeing new connections to other things, that is a powerful idea. You're not just learning one thing, but learning how to learn new things. Since this happening, other ways of equal interest were invented by other people; Brad's way just happened to end up as an infinite series.

Can you find a simple fraction for Brad's infinite series? It looks very much like the first series we talked about:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots = 1$. Brad's series is this one but we have to take out $\frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \ldots$ or $\frac{1}{8} + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \ldots$ So Brad's series $= 1 - \left[\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \ldots\right]$. From our earlier work, the infinite series in the brackets $= \frac{1}{8 - 1} = \frac{1}{7}$; Brad's series $= 1 - \frac{1}{7} = \frac{6}{7}$.

The following are other solutions to Brad's problem, given each share, as done by a variety of people of different ages: $\frac{1}{28} + \frac{1}{14} + \frac{1}{4} + \frac{1}{2}$, by Leslie, age 10; $\frac{1}{2} + \frac{2}{6} + \frac{1}{42}$, by Cindy; $\frac{3}{4} + 1$ pica $+ \frac{1}{4}$ of a pica, by an adult typesetter; $\frac{2}{3} + \left(\frac{4}{7}\right) \cdot \left(\frac{1}{3}\right)$, by a principal of a middle school; $\frac{5}{6} + \frac{1}{42}$, by an elementary school principal. Most people of say 6th grade and up, however, said they got $\frac{6}{7}$ by cutting the 6 cookies into 7 pieces each, then each person gets $\frac{1}{7}$, 6 times or $\frac{6}{7}$. For young people, cutting into 7 pieces is not a normal thing, and is difficult; cutting a thing in half, into 2 pieces, is easy and familiar.

An interesting offshoot of Brad's method is that his series can be written as an infinite repeating bimal .11011011 . . . equal to $\frac{6}{7}$ (or in binary $\frac{110}{111}$), in which the places are $\frac{1}{2}$'s, $\frac{1}{4}$'s, $\frac{1}{8}$'s and the numerators are those in the series, instead of $\frac{1}{10}$'s, $\frac{1}{100}$'s, $\frac{1}{1000}$'s etc., in our regular decimal system. $\frac{6}{7}$ as an infinite repeating decimal is .857142857142 . . .

It's interesting to look at these numbers for patterns—I just saw the 110 in the fraction $\frac{110}{111}$ and 110 is repeated in the bimal .110110 . . . And $\frac{43}{99}$ = .434343 . . . in decimal.

CHAPTER 3: Ian's Proof: Infinity = $^-1$

At age 7 Ian Robertson was generalizing the area of a triangle, realizing his formula would work for all triangles. That to me had the flavor of an infinite number of things. When he was 11 years old, in January 1982, he came back from winter vacation with The Mathematics Calendar for 1982 (see bibliography). He had solved the infinite problem:

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}} = 3$$

He argued, what's under the biggest square root is 9, because $\sqrt{9} = 3$. He sets what's under the radical then, equal to 9.

So

$$\underbrace{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}_{\text{But this much is the original problem and equals 3}} = 9$$

So $x + 3 = 9$ and $x = 6$.

Ian generalized these infinite square roots in September of '82. He worked on infinite continued fractions and starting in January 1982 he was reading Sawyer's "What Is Calculus About".

Ian was clearly thinking about derivatives, and by November of '82 did $\frac{f(x + h) - f(x)}{x + h - x}$ for $f(x) = x^2$. In '83 he sees he can work derivatives backwards and forwards (see chapter 14).

In November of 1983 Ian wrote down how he figured out the sum of the infinite series $1 + a + a^2 + a^3 + a^4 + \ldots = \frac{1}{1 - a}$, the same one Euler had done in 1754–55. It went simply like this:

$$1 + a + a^2 + a^3 + a^4 + \ldots = C \quad \text{he factored out } a$$

$$1 + a(1 + a + a^2 + a^3 + \ldots) = C$$

$$1 + a\underbrace{(1 + a(1 + a(1 + \ldots)))}_{\text{this is C}} = C \quad \text{similar to infinite } \sqrt{\ }$$

$$1 + aC = C \quad \text{subtract C and add } {}^{-}1$$

$$aC - C = {}^{-}1 \qquad \text{factor out C}$$

$C(a - 1) = {}^{-}1$ divide by $a - 1$

$C = \frac{-1}{a - 1} = \frac{1}{1 - a}$ by multiplying top and bottom by ${}^{-}1$.

So $1 + a + a^2 + a^3 + a^4 + \ldots = \frac{1}{1 - a}$

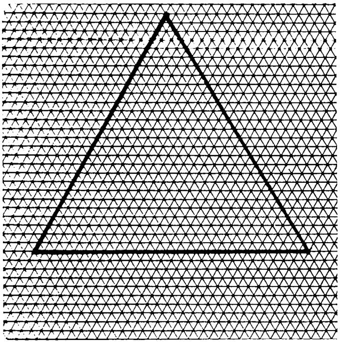
Now he argued,

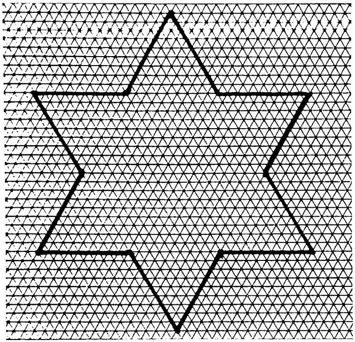
if $a = 2$, then $1 + 2 + 4 + 8 + 16 + \ldots = \frac{1}{1 - 2} = {}^{-}1$, and since $1 + 2 + 4 + 8 + 16 + \ldots$ goes to infinity,

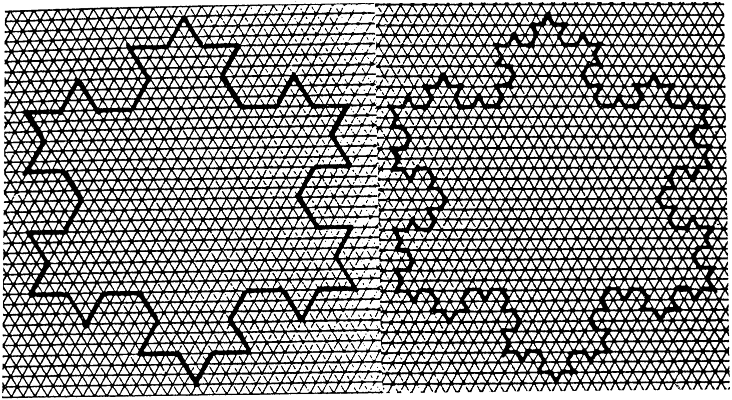
$$\infty = {}^{-}1$$

CHAPTER 4: The Snowflake Curve—its Area and Perimeter

To build the snowflake curve you start with an equilateral triangle. The succeeding figures are made by dividing each side into 3 equal parts then adding a triangular piece on each of the center pieces of the sides. The first four are shown below:







The first problem is to find the area of each figure and then tell what happens to the area if this process goes on forever.

I recommend calling the area of the first figure one unit of area. You might want to work on this yourself before reading on.

In finding the area of the snowflake we get the following sequence of partial sums (call the area of the first figure one):

1

$1 + \frac{3}{9}$ notice we're not writing one number for the answer; you want an answer to help see a pattern. This is important. In elementary schools the emphasis is usually only on getting the immediate answer rather than on patterns that will get any and all answers, a generalization. The answer here is better not left as a number, but as a sum, then finding a pattern for the infinite series:

$$1 + \frac{3}{9} + \frac{12}{81}$$

$$1 + \frac{3}{9} + \frac{12}{81} + \frac{48}{729}$$

Now comes a tricky part, trying to figure how to write these fractions to see a pattern; one way which begins to work is:

$$1 + 3 \cdot \frac{4^0}{9^1} + 3 \cdot \frac{4^1}{9^2} + 3 \cdot \frac{4^2}{9^3} + \ \ldots$$

Now we'd like to get an infinite series to look like one in Chapter 1, which we can figure out. Forget the 1 for a moment. Factor out 3 from each term after the 1.

$$1 + 3 \cdot \left(\frac{4^0}{9^1} + \frac{4^1}{9^2} + \frac{4^2}{9^3} + \ldots\right)$$

This is still hard because the exponents of 4 and 9 are not the same. To fix this, multiply each term inside the parentheses by 4 and divide by 4 on the outside of the parentheses. This doesn't change things because we're multiplying by 1 as $\frac{4}{4}$. We end up with the same exponents for 4 and 9:

$$1 + \frac{3}{4} \cdot \left[\frac{4^1}{9^1} + \frac{4^2}{9^2} + \frac{4^3}{9^3} + \ldots\right]$$

The series in the brackets is like one of our earlier ones and goes to $\frac{4}{9 - 4}$ or $\frac{4}{5}$.

Then the area of the snowflake converges to $1 + \frac{3}{4} \cdot \frac{4}{5} = 1\frac{3}{5}$.

It's fascinating to see 8- and 9-year-olds do these things!

Now see if you can find the perimeter. Call the perimeter of the first one 1 if you like, as I did. That's not essential.

STOP HERE AND TRY IT. LOOK AT THE DIAGRAMS ABOVE.

Finding the perimeter goes like this:

$$1$$

$$1 + \frac{1}{3}$$

$$1 + \frac{1}{3} + \frac{12}{27}$$

$$1 + \frac{1}{3} + \frac{12}{27} + \frac{48}{81}$$

again we have the problem of how to write these terms; it takes time and some mistakes before it becomes clear. We'll write it this way:

$$1 + \frac{1}{3} + \frac{4 \cdot 3}{3^3} + \frac{16 \cdot 3}{3^4} + \ \ldots \text{ or}$$

$$1 + \frac{2^0}{3^1} + \frac{2^2}{3^2} + \frac{2^4}{3^3} + \ \ldots$$ We're getting closer.

Remember, there are different ways to do this.
How about this:

$$1 + \frac{4^0}{3^1} + \frac{4^1}{3^2} + \frac{4^2}{3^3} + \ \ldots \text{ now we can factor out } \frac{1}{3}$$

$$1 + \frac{1}{3} \cdot \left(\frac{4^0}{3^0} + \frac{4^1}{3^1} + \frac{4^2}{3^2} + \ \ldots \right)$$

$$1 + \frac{1}{3} \cdot \left[\left(\frac{4}{3}\right)^0 + \left(\frac{4}{3}\right)^1 + \left(\frac{4}{3}\right)^2 + \ \ldots \right]$$

Remember when we did the original ones and found they go to a number if $A < B$. In this case $A > B$, $4 > 3$ and $\frac{4}{3}$ is bigger than 1, so it keeps getting bigger and bigger and we say it is divergent.

So here is this snowflake curve, the area is a convergent series and goes to $1\frac{3}{5}$, while its perimeter is a divergent series and goes to infinity. Very interesting.

This curve is sometimes called one of the "pathological" curves, compared to simple straight lines, parabolas, circles, ellipses, hyperbolas and sine waves, which Kasner and Newman describe as "healthy and normal". In fact today the snowflake curve, one of the Koch curves, is in the realm of Fractal Geometry and part of chaos theory. Mandelbrot, the leading force in this new way of looking at geometry, talks about coastlines, turbulence, clouds, galaxies and tries to describe the irregular and fragmented in Nature, with applications in economics as well as many other fields. Imbedded in his work is the idea of self-similarity.

CHAPTER 5: The Harmonic Series

Take a look at this infinite series:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots$$

which is called the harmonic series. See what you can find out about it.

The harmonic series looks innocent enough. My students started to use their calculators and even wrote computer programs on the FX7000G to look at the sums. One day some youngsters must have gotten the sum of the first 10,000 terms, and it didn't reach 10! They were ready to say the infinite sum went to 10 as the limit. Karl T. Cooper wrote from Providence, R.I., that using his computer, he found the sum of the first 1,000,000 terms is 13.3573617935, not 14.392726788474, as I incorrectly reported in earlier printings. He showed this graphically as well. This series tends to infinity, diverges, but very slowly. It's wonderful to hear from people who are really reading my books. No matter how many times I've checked, there are still mistakes. It's nice to know I'm not perfect!

This series tends to infinity, diverges, but very slowly.

In about 1350 Oresme proved this series to be divergent. The proof goes like this: we'll compare the harmonic series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\dots$$

to this series, $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\dots$

which gets bigger and bigger and is divergent. We'll show that the terms of the harmonic series, looked at in a certain way, will be bigger than the terms of this series $\frac{1}{2}+\frac{1}{2}+\dots$ and therefore is divergent also. The first terms, $\frac{1}{2}$, are the same. Since $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$, and $\frac{1}{3}>\frac{1}{4}$, then $\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$. The second 2 terms of the harmonic series add up to a number bigger than $\frac{1}{2}$.

Now $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$, but $\frac{1}{5} > \frac{1}{8}$, $\frac{1}{6} > \frac{1}{8}$ and $\frac{1}{7} > \frac{1}{8}$ So the sum of the next 4 terms of the harmonic series $\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$ is bigger than $\frac{1}{2}$ also. The next 8 terms will be bigger than $\frac{1}{2}$ and so on. Since this is an infinite sum, we can go on and on and the harmonic series will be bigger than the series $\frac{1}{2} + \frac{1}{2} + \ldots$ and is therefore divergent also.

CHAPTER 6: On Thin Spaghetti and Nocturnal Animals

A group of teachers at a National Association of Independent Schools workshop in 1975, spent time on guessing functions and graphing functions and came out with some interesting results, including an infinite sequence.

It's important enough to digress a moment now to do the guess my rule or guessing functions: I'm thinking of a machine or rule. You give me a number (input), I put your number in my machine and give you a number back (output). I always do the same thing to the number you give me. Your job is to figure out how my rule works. So if you give me 1, I tell you 5; if you tell me 2, I tell you 7, and so on. We'll put the numbers in a table like that at the right. Can you guess my rule?

input (x)	output (y)
1	5
2	7
3	9
4	11
10	23

There are many ways to say it. If your way gets the same numbers, that's fine. Some people would say I added your the number twice then add 3. Others might say I doubled your number then count up 3. Let's write the rule different ways using the x and y. One way to write it is $x + x + 3 = y$, another is $2 \cdot x + 3 = y$. Notice, we're doing arithmetic and algebra together, and for young people this is easy and fun.

Now you make up a rule for me. Use a 3×5 card, put your table of numbers and name on one side and your written rule on the other side. Here are some rules you can try to figure out:

1.

x	y
0	$^{-}3$
1	0
2	3
3	6
4	9

2.

x	y
0	2
1	7
2	12
3	17
4	22

3.

x	y
0	0
1	1
2	3
3	6
4	10

4.

x	y
0	0
1	1
2	4
3	9
4	16

5.

x	y
1	1
2	5
3	14
4	30
5	55

6.

x	y
0	1
1	2
2	4
3	8
4	16

7.

x	y
2	180
3	120
4	90
5	72
6	60

8.

x	y
212	100
122	50
77	25
68	20
32	0

The other idea which is very important is the graphing of equations and these functions when we play guess my rule. I start this with 5-year-olds right away because it involves simple arithmetic, simple algebra and simple geometry, all at the same time. With the 5's I start with an equation like $x + y = 7$.

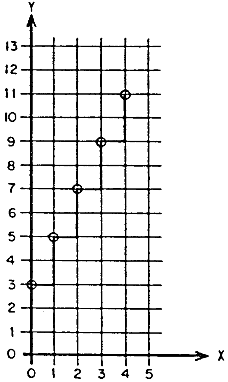
This will give a picture of the pairs of numbers that add to 7. *Each pair of numbers corresponds to a point on the graph,* where the lines cross.

If we graph the pairs of numbers from the rule $2 \cdot x + 3 = y$,

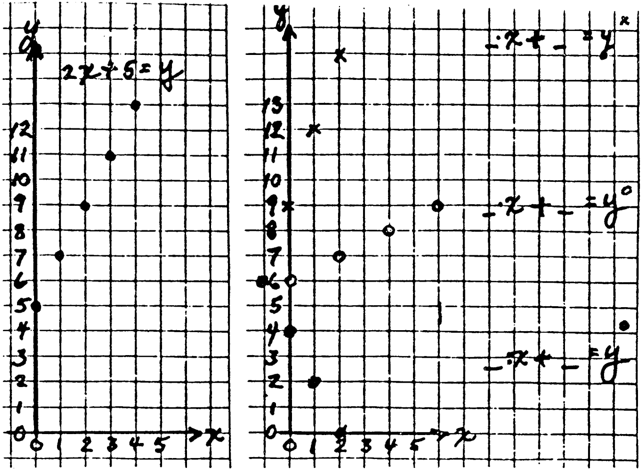
it looks like this:

$2 \cdot x + 3 = y$

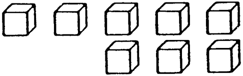
x	y
1	5
2	7
3	9
4	11



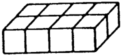
How is the graph on the left the same as/different from, $2 \cdot x + 3 = y$? Find the equations for the following graphs:



Back to the teachers' work. We looked at the problem: given a certain volume, like 8 cubic centimeters of wood, find their surface or skin area. The Cuisenaire rods we used are $1 \times 1 \times 1$ to $1 \times 1 \times 10$ cm. We found the following arrangements:

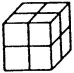


S. A. = 48 cm^2

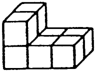


S. A. = 28 cm^2

S. A. = 34 cm^2



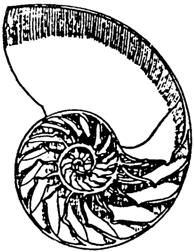
S. A. = 24 cm^2



S. A. = 28 cm^2

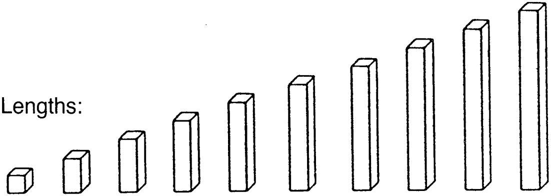
We noticed that the smallest surface area, for a given volume, occurred when we built a cube! This same idea occurs in spherical soap bubbles.

We looked at the Nautilus shell. One of the teachers concluded that the distance across each chamber does not increase as fast as the volume of each chamber. Since that time, various people have done studies of the shell and how it grows, which we will not include here.



This led to a study of how lengths, squares and cubes grow.

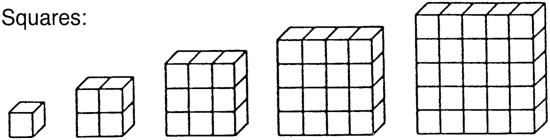
Lengths:



X ROD	Y LENGTH
1	1
2	2
3	3
4	4
5	5
⋮	⋮

RULE: $X = Y$

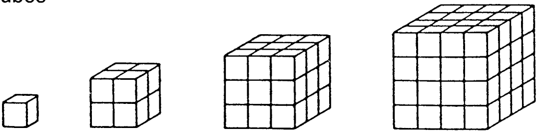
Squares:



X LENGTH OF SIDE	Y AREA OF SQUARE
1	1
2	4
3	9
4	16
5	25
⋮	⋮

RULE: $X^2 = Y$

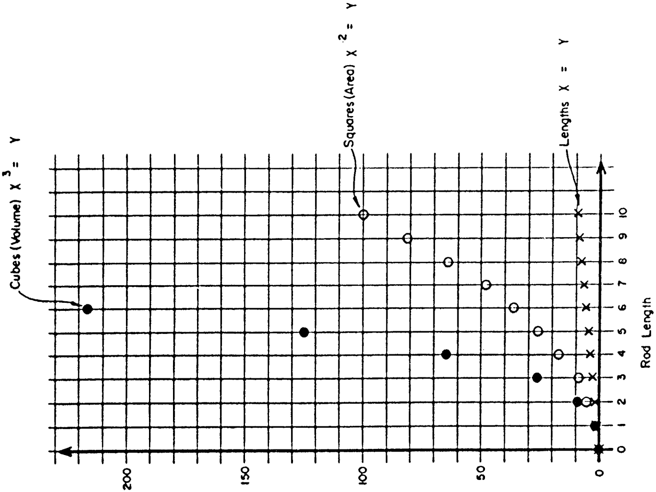
Cubes



RULE: $X^3 = Y$

X LENGTH OF EDGE	Y VOLUME OF CUBE
1	1
2	8
3	27
4	64
5	125
⋮	⋮

We graphed this data:



One of the teachers suggested looking at the surface area of cubes, not the single rods, because the single rods have the same cross section of 1 cm^2. From the data below, the surface area goes up as the square of the length.

Length (l)	Surface Area
1	$1^2 \times 6$
2	$2^2 \times 6$
3	$3^2 \times 6$
4	$4^2 \times 6$
5	$5^2 \times 6$

Surface Area = $l^2 \times 6$

The volume of the cubes goes up as the cube of the length as seen from the data on the cubes.
We then looked at the surface area to volume ratio of the rods:

Length	Surface Area	Volume	$\frac{\text{Surface Area}}{\text{Volume}}$
1	6	1	$\frac{6}{1} = 6$
2	10	2	$\frac{10}{2} = 5$
3	14	3	$\frac{14}{3} = 4.67$
4	18	4	$\frac{18}{4} = 4.5$
5	22	5	$\frac{22}{5} = 4.4$
6	26	6	$\frac{26}{6} = 4.33$
7	30	7	$\frac{30}{7} = 4.28$
8	34	8	$\frac{34}{8} = 4.25$
9	38	9	$\frac{38}{9} = 4.22$
10	42	10	$\frac{42}{10} = 4.2$
⋮	⋮	⋮	⋮
100	402	100	$\frac{402}{100} = 4.02$
⋮	⋮	⋮	⋮
1000	4002	1000	$\frac{4002}{1000} = 4.002$
⋮	⋮	⋮	⋮
$\square$	$4 \times \square + 2$	$\square$	$\frac{4 \times \square + 2}{\square} = \frac{4 \times \square}{\square} + \frac{2}{\square} = 4 + \frac{2}{\square}$

For the $\frac{\text{S.A.}}{\text{Vol.}}$ ratio we get the infinite sequence

6, 5, 4.67, 4.5, 4.4, 4.33, 4.28, 4.25, 4.22, . . .

which decreases, gets closer and closer to 4 and never gets below 4. The limit of this sequence is 4.

If the white rod ($1 \times 1 \times 1$) is a mouse, and the orange rod ($1 \times 1 \times 10$) a human, the mouse has a greater surface area to volume ratio. The skin acts to rid the body of perspiration and the mass (proportional to the volume) is the measure of heat production. If the mouse ran around during a sunny day, it would lead to "excessive transpiration". That's why rodents are nocturnal animals. A visitor came in on the discussion to say "that's why mice eat more for their weight than elephants and why my sons like thin spaghetti because there is more surface area to be surrounded by sauce than thick spaghetti!" And why we grate cheese before putting it on the spaghetti.

Try to guess a rule for each of these investigations and graph your data:

1. How many squares can you make on a 4 $\times$ 4 array of dots? in a 5 $\times$ 5? in a 20 $\times$ 20? (Vertices at dots).
2. For six equally spaced dots on a circle, how many straight line segments connect these points?
3. # of diagonals of a polygon vs. the number of sides.
4. The sum of the interior angles of a polygon vs. the number of sides.
5. The interior angle of a polygon vs. the number of sides
6. The central angle of regular polygon vs. the number of sides.
7. Rectangles with a constant perimeter of 20; length vs. width. } See
8. Rectangles with a constant perimeter of 20; length vs. area. } Chapter 14
9. Rectangles with a constant area of 36; length vs. width.
10. Rectangles with a constant area of 36; length vs. perimeter.
11. Side of a square vs. perimeter
12. Side of a square vs. area.

13. Edge of a cube vs. surface area.
14. Edge of cube vs. volume.
15. Celsius vs. Fahrenheit temperatures.
16. Weight vs. stretch of a spring.
17. Height of an object vs. the length of its shadow.
18. Weight vs. value of the same coins.
19. # of straight lines vs. maximum # of intersections.
20. Weight vs. volume of various size pieces of the same solid.
21. # of wheel turns vs. # of pedal turns on a bicycle.
22. Distance the wheel moves vs. # of pedal turns on a bicycle.
23. # of teeth on a gear vs. its diameter.
24. Perimeter vs. diameter of various circular objects.
25. Tower puzzle: # discs vs. minimum # of moves.
26. Shuttle puzzle: # of pairs of pegs vs. # of moves.
27. Angle of sun vs. day.
28. # of hours of sunlight vs. day
29. Height of ball vs. height of bounce.
30. # of Kwh vs. cost of electricity.

31. oz. vs. gm. on cans or boxes of food.
32. # of miles travelled vs. # of gallons of gas consumed.
33. # of gallons of gas purchased vs. total purchase price.
34. Length of a pendulum (string with weight) vs. time for ten swings.
35. Length of wire vs. its resistance.
36. Height of a candle vs. time to burn.
37. Length of the chord of a circle vs. the # of degrees in the smaller arc.

CHAPTER 7: The Fibonacci Numbers, Pineapples, Sunflowers and The Golden Mean

The Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34 . . . add the last two numbers to get the next number. This is an infinite sequence that is divergent. If we take the ratio of each number to the number before it we get another infinite sequence:

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots$ Looking at the mixed fractions and decimals, there are many patterns:

$$\frac{1}{1} = 1 = 1.00000 \qquad \frac{2}{1} = 2 = 2.00000$$

$$\frac{3}{2} = 1\frac{1}{2} = 1.50000 \qquad \frac{5}{3} = 1\frac{2}{3} = 1.66666\ldots$$

$$\frac{8}{5} = 1\frac{3}{5} = 1.60000 \qquad \frac{13}{8} = 1\frac{5}{8} = 1.62500$$

$$\frac{21}{13} = 1\frac{8}{13} = 1.61538\ldots \qquad \frac{34}{21} = 1\frac{13}{21} = 1.61904\ldots$$

$$\frac{55}{34} = 1\frac{21}{34} = 1.61764\ldots \qquad \frac{89}{55} = 1\frac{34}{55} = 1.61818\ldots$$

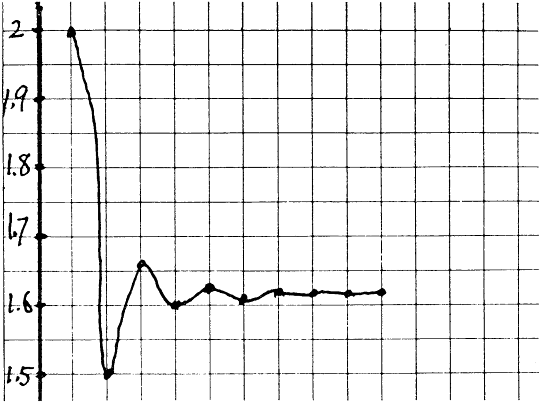
$$\frac{144}{89} = 1\frac{55}{89} = 1.61797\ldots \qquad \frac{233}{144} = 1\frac{89}{144} = 1.61805\ldots$$

$$\frac{377}{233} = 1\frac{144}{233} = 1.61802\ldots \qquad \frac{610}{377} = 1\frac{233}{377} = 1.61803\ldots$$

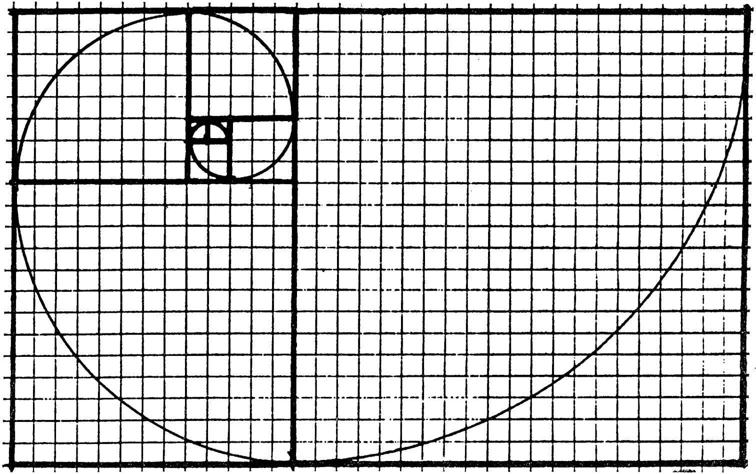
What patterns do you see?

The numerators in the ratios are the original sequence. as are the denominators but one behind. The mixed fractions act similarly. The decimals form an alternating sequence, they increase then decrease then increase, etc., but they are getting closer to some number like 1.61803 . . . This is an irrational number $\frac{1 + \sqrt{5}}{2}$ and is special, given the name The Golden Mean or The Divine Proportion.

Below is a graph that has been made by many young people, showing the infinite sequence of ratios of Fibonacci numbers that we saw above.



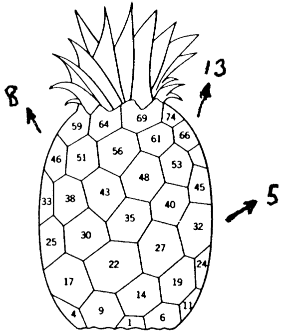
The Spiral can be obtained from building squares onto rectangles, using the Fibonacci numbers, as in the figure below:



The spiral is made up of arcs of circles whose centers are at the corners of the squares.

The Fibonacci numbers are found in the leaf arrangement (phyllotaxis) of various plants. Count the rows of the 2 sets of nearly hexagonal cells of a pineapple as they whorl around it. The pineapples I bought had 8 and 13 rows, both numbers being Fibonacci numbers. Coxeter says there is another set of whorling cells of 5 rows; that took longer to find.

Many of us have counted the leaves on the stem of a sunflower plant as they go around and up to a leaf above a previous one; this number came out to be 8. We've counted the number of times the leaves



whorl around the branch and is turned out to be 3. Both 3 and 8 are Fibonacci numbers.

The golden angle (about 137.5°) was obtained by a group of teachers examining the sunflower leaves, then finding fractions of 360° using alternate Fibonacci number ratios:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 . . .

$$\frac{1}{2} \times 360 = 180 \text{ degrees} \qquad \frac{1}{3} \times 360 = 120$$

$$\frac{2}{5} \times 360 = 144 \qquad \frac{3}{8} \times 360 = 135$$

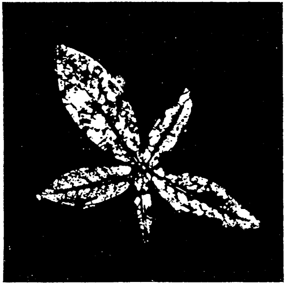
$$\frac{5}{13} \times 360 = 138.46\ldots \qquad \frac{8}{21} \times 360 = 137.14\ldots$$

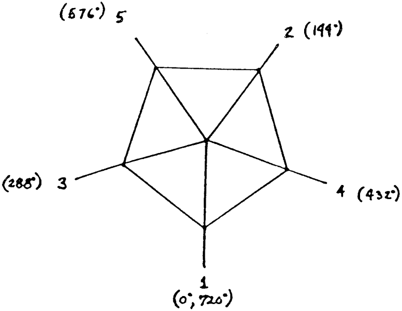
$$\frac{13}{34} \times 360 = 137.64\ldots \qquad \frac{21}{55} \times 360 = 137.46\ldots$$

$$\frac{34}{89} \times 360 = 137.52\ldots \qquad \frac{55}{144} \times 360 = 137.50$$

This gives us an alternating sequence whose limit is 137.50 . . . (irrational) and can be written as $(.618034\ldots)^2 \times 360$ or $\frac{3 - \sqrt{5}}{2}$ times 360. This angle of 137.5° or 137°30'28", is the angle that allows each leaf to be closest to the leaf below it in the previous whorl and farthest from the youngest previous leaf, in other words, it allows the leaf to get maximum sunlight.

The second way to use the sunflower is to count the spirals of sunflower seeds on the head of the plant. There are two sets of whorling seeds; the number of these in each set varies upon the size of the sunflower, but each number is a Fibonacci number (or close to it). We have gotten 55 and 89. Counting these spiralling rows is tricky; we put small colored pins in each row, then counted the pins. Since nothing in nature is perfect, the rows are difficult to count. Whoever said counting is simple, obviously never tried to count sunflower seeds or rows on a pineapple!

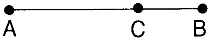




An example of a plant whose leaves whorl at a angle of 144°.

I have found bushes near the house which have a 3, 8 arrangement of leaves. It took me quite a long time, a year or two, struggling to find these leaf arrangements- like everything else, it takes persistence!

One of the more common methods of getting the golden mean or golden section is to cut a line segment AB at point C such that the following proportion works:



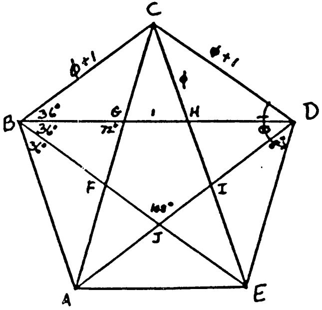
$$\frac{\text{the whole segment AB}}{\text{larger segment AC}} = \frac{\text{larger segment AC}}{\text{smaller segment CB}}$$

If we call the larger segment x, the smaller segment, 1, and the whole segment $x + 1$, then we get the equation

$$\frac{x+1}{x} = \frac{x}{1} \quad \text{or} \quad x^2 = x + 1 \text{ or } x^2 - x - 1 = 0$$

We can solve the $x^2 = x + 1$ version with infinite continued fractions as in CHAPTER 8. Barb and Jenny used the quadratic formula below.

Barbara and Jenny, both 9th graders, both long-timers in The Math Program, started with a regular pentagon, drew its diagonals, then figured out all the angles in the figure on the right.



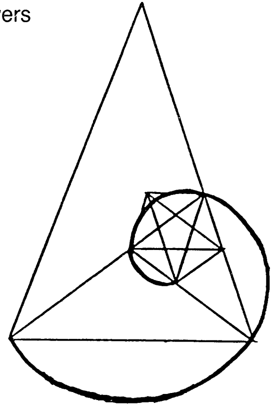
They figured out that there are only 3 different angles in the figure 36°, 72° (2 $\times$ 36) and 108° (3 $\times$ 36). We found $\frac{\sin 72^\circ}{\sin 36^\circ} = \phi = 1.6$; $\frac{\sin 108^\circ}{\sin 36^\circ} = \phi$ and $\frac{\sin 108^\circ}{\sin 72^\circ} = 1$. Triangle GCH, the 36°-72°-72° triangle is a golden triangle, because $\frac{CG}{GH} = \frac{CH}{GH} = 1.6 = \phi$ from measuring the lengths. Triangle HBC is similar to triangle GCH and also a golden triangle. Therefore the following proportion must be true $\frac{\phi}{1} = \frac{\phi + 1}{\phi}$ and $\phi^2 = \phi + 1$. $\phi + 1$ is also the length of the side of the pentagon. Triangle GCH is similar to triangle DAC, so the following proportion is true:

$\frac{\phi}{1} = \frac{2\phi + 1}{\phi + 1} = \frac{2\phi + 1}{\phi^2}$ and $\phi^3 = 2\phi + 1$. Using larger triangles they found $\phi^4 = 3\phi + 2$ and $\phi^5 = 5\phi + 3$. They came out with the quadratic equation $\phi^2 - \phi - 1 = 0$, from the one above and solved it with the quadratic formula:

$$x = \frac{{}^{-}b \pm \sqrt{b^2 - 4ac}}{2a} \text{ with } a = 1,\ b = {}^{-}1 \text{ and } c = {}^{-}1$$

$\phi_1 = \frac{1 + \sqrt{5}}{2}$ and $\phi_2 = \frac{1 - \sqrt{5}}{2}$. They realized that $\phi_1 \cdot \phi_2 = {}^{-}1$ and $\phi_1 + \phi_2 = 1$.

The golden spiral came out as shown below. They also saw a pattern in the powers of ϕ and wrote a program to print it out:



$$
\begin{aligned}
\phi^{-5} &= 5 \cdot \phi + {}^{-}8 \\
\phi^{-4} &= {}^{-}3 \cdot \phi + 5 \\
\phi^{-1} &= 2 \cdot \phi + {}^{-}3 \\
\phi^{-2} &= {}^{-}1 \cdot \phi + 2 \\
\phi^{-1} &= 1 \cdot \phi + {}^{-}1 \\
\phi^{0} &= 0 \cdot \phi + 1 \\
\phi^{1} &= 1 \cdot \phi + 0 \\
\phi^{2} &= 1 \cdot \phi + 1 \\
\phi^{3} &= 2 \cdot \phi + 1 \\
\phi^{4} &= 3 \cdot \phi + 2 \\
\phi^{5} &= 5 \cdot \phi + 3 \\
\phi^{6} &= 8 \cdot \phi + 5
\end{aligned}
$$

. . . AND YOU SHALL MEET A HORRIBLE FATE . . . YOU SHALL SPEND ALL ETERNITY FINDING POWERS OF PHI . . .

CHAPTER 8: Solving Equations/Infinite Continued Fractions

Let's change the improper fraction $\frac{37}{28}$ into a finite continued fraction.

$$\frac{37}{28} = 1 + \frac{9}{28} = 1 + \frac{1}{\frac{28}{9}} = 1 + \cfrac{1}{3 + \cfrac{1}{9}}$$

This is as far as we can go (until, when we divide the last reciprocal, the remainder is 0, $\frac{9}{1} = 9$ remainder 0). Change this to a continued fraction: $\frac{43}{18}$.

Change the following continued fraction to an improper fraction:

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{7}}}$$

We'll be using infinite continued fractions in the rest of this chapter because we will be representing irrational or transcendental numbers.

How many ways can you solve this quadratic equation $x^2 - 5x + 6 = 0$?

My first interest in infinite continued fractions came from looking at different methods to solve quadratic equations. I'm up to about 12 ways to do that now, seven of which are shown below. Then I found I could write $\sqrt{2}$, the golden mean (ϕ) and π as infinite continued fractions. Also, there are so many patterns within them. The infinite continued fraction gives one the ability to find an infinite sequence of approximations that converges and we can program a computer to do this.

METHOD 1 to solve a quadratic equation:

	Solve	$x^2 - 5x + 6 = 0$	add 5x and $^-6$
		$x^2 = 5x - 6$	divide by x
Eq. 1		$x = 5 - \frac{6}{x}$	

Now comes the interesting part. Since a name for x is $5 - \frac{6}{x}$, that's what the Eq.1 above says, we can substitute $5 - \frac{6}{x}$ in for x, on the right side. We get

Eq. 2 $$x = 5 - \cfrac{6}{5 - \cfrac{6}{x}}$$ and then just continue that

Eq. 3 $$x = 5 - \cfrac{6}{5 - \cfrac{6}{5 - \cfrac{6}{x}}}$$

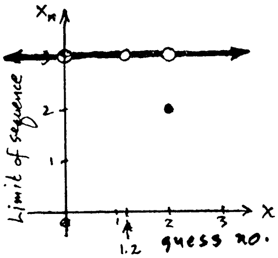
Eq. 4 $$x = 5 - \cfrac{6}{5 - \cfrac{6}{5 - \cfrac{6}{5 - \cfrac{6}{x}}}}$$ . . . and so on.

This is an infinite continued fraction, unusual, interesting. There are a couple of ways to work with these. One way is to guess a number for x say 1, put it in for x on the right side in Eq.1., get an answer $^-1$, this will be the first approximation. Then put the same guess, 1, in the right side of Eq.2, which gives the second approximation, and so on. This gives an infinite sequence of rational numbers (out to 3 decimal places and 4 after a while) $^-1$, 11, 4.455, 3.653, 3.358, 3.213, 3.133, 3.085, 3.055, 3.036, 3.024, 3.016, 3.010, 3.007, 3.005, 3.003, 3.002, 3.001, 3.0009, 3.0006, 3.0004, and it gets closer and closer to 3.

The other way to look at these, and a simpler way, is to just use Eq.1. Put the guess number, 1, in for x on the right side. Then take the answer, $^-1$, put it in for x on the right side in Eq.1 again. The answer is 11. Keep doing that and we obtain the same sequence as we did above. We can write a simple computer program to do this in basic (to get 20 terms): 10 N $=$ 0; 20 input x; 30 x $=$ $5 - \frac{6}{x}$; 40 PRINT x; 50 N $=$ N $+$ 1; 60 IF N $<$ 20 THEN GOTO 30;
70 STOP.

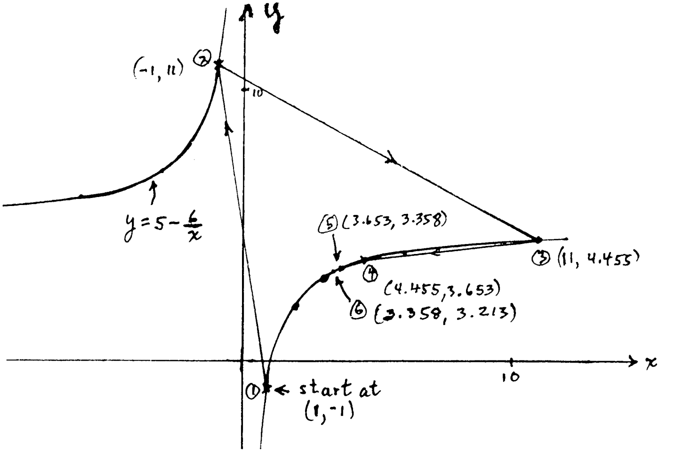
The program for the FX7000G is: ? → x: Lbl 5: 5 $- \frac{6}{x} \rightarrow x$◢ Goto 5.

If you graph the guess number vs. the number the sequence goes to, its limit, it is very interesting. Notice on the graph 2 goes directly to 2 and 3 goes directly to 3. An infinite number of guess numbers you try will form a sequence that goes to 3. *But this graph is wrong.* We found not only 0 (shown), but $\frac{6}{5}$, $\frac{30}{19}$ and an infinite number of numbers (obtained by putting 0 then $\frac{6}{5}$ etc. in the left side),



make it blow up, because eventually you get 0 in the denominator. So there are an infinite number of holes missing on the graph.

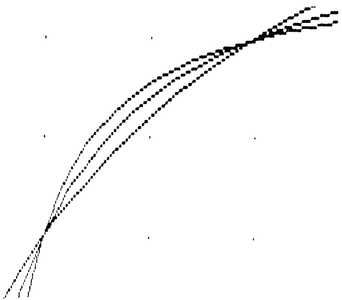
METHOD 2: GRAPH $y = 5 - \frac{6}{x}$, choose a guess number, say 1 for x (as in method 1 above), then show the infinite sequence approaching 3 as it moves along the curve. The points are pairs of consecutive numbers in that sequence $(1,^-1)$, $(^-1,11)$, $(11,4.45)$, . . .



METHOD 3: GRAPH $y = 5 - \frac{6}{x}$,

then $y = 5 - \dfrac{6}{5 - \frac{6}{x}}$ and

$$y = 5 - \cfrac{6}{5 - \cfrac{6}{5 - \frac{6}{x}}}.$$



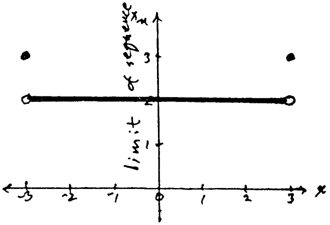
We get 3 hyperbolas and much to our amazement they intersect at the points (2,2) and (3,3), the two solutions of our quadratic equation $x^2 - 5x + 6 = 0$! In the graph at the right you only see one piece of each hyperbola, and only the section where they intersect.

The nice thing about all this is that something new is happening all the time—if you are willing to let it, willing to learn new things and willing to say you don't know everything. As Sr. Jacqueline, at that time president of Webster College said, "we have to learn to be secure in our insecurities".

METHOD 4: Let's try the same equation again, this time solving it a different way:

$x^2 - 5x + 6 = 0$; add $5x \rightarrow 5x = x^2 + 6$; divide by 5

Eq.5 $\qquad x = \dfrac{x^2 + 6}{5}$ We can now do a similar thing as we did above, put the whole right side in for x on the right side. OR, we'll just write a program to put in a guess number for x, do the calculation, then put the new approximation back in for x again in Eq.5. The graph of this solution is shown at the right.

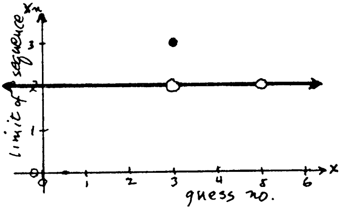


Again some interesting things happen. For 3 or $^-3$ it goes directly to 3. For any guess number whose absolute value is >3 we get a diverging sequence. For any guess whose absolute value is <3 we get an infinite sequence which converges to 2.

METHOD 5: $x^2 - 5x + 6 = 0$ add $^-6$, factor left side

$x(x - 5) = {}^-6$ $\div (x - 5)$

Eq. 6 $$x = \frac{-6}{x - 5}$$ Here's another infinite



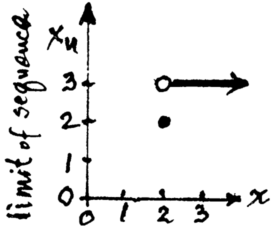
cont'd fraction. *The incorrect graph is at the right.* 3 goes directly to 3. For an infinite number of guess numbers we get infinite sequences that converge to 2. But for x = 5 (shown), $\frac{19}{5}, \frac{65}{19}$ and an infinite number of numbers (obtained by putting 5, then $\frac{19}{5}$, etc in the left side) the denominator is 0 and there is no solution (holes not shown).

METHOD 6: $x^2 - 5x + 6 = 0$ add 5x and $^{-}6$

$x^2 = 5x - 6$ take $\sqrt{}$ of both sides

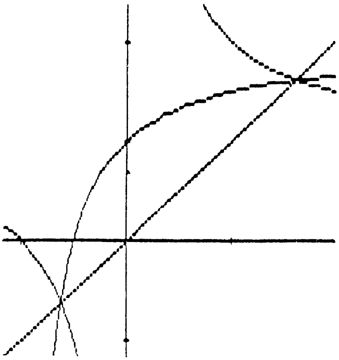
Eq. 7 $x = \sqrt{5x - 6}$ This is different. It is not

an infinite continued fraction, but an infinite continued radical something like Ian had (see Chapter 3).
The method is the same as above; just put the radical in for *x* on the right side. The graph of this is at the right, using just the positive radicals. Every guess number less than 2 goes imaginary. 2 goes directly to 2, and all guesses >2 go to 3.



METHOD 7: Jeff, a 5th-grader, knew the roots of the equation $x^2 - 5x + 6 = 0$ were 3 and 2 very quickly, because $3 + 2 = 5$ and $3 \times 2 = 6$. He remembered having done that on Plato last year (Jerry and I did that work 12 years ago on Plato, the computer-based education project, directed by Don Bitzer at the U of I). He also got the roots of $x^2 - 25x + 24 = 0$, 24 and 1, right away. I decided he needed a hard one, so I gave him $x^2 - x - 1 = 0$. He tried 1 which gave $^-1$ for the left side and of course, not equal to 0. At this point he reached for a calculator and started using decimals, like 1.5 and 1.6. Jeff got a sequence of numbers too big and too small. He ended up with 1.618034 to give him 0. He thought about this, then said "I suppose there must be another number." I thought that was great and told him so. He thought about what the other number should be and said it should be $^-.618034$. He figured that the coefficient of x was $^-1$, so he subtracted 1 from 1.618034 and made it negative. And of course he was right. I then told him that this number was the Golden Mean and how the ancient Greeks used that ratio to build the Parthenon.

We got the infinite continued fraction for ϕ from the equation $x^2 - x - 1 = 0$. We added x and 1 to both to get $x^2 = x + 1$. Then divided by x to get $x = 1 + \frac{1}{x}$. We then put $1 + \frac{1}{x}$ in for x on the right side of the equation and continued this process. On the way we graphed (at the right), $y = 1 + \frac{1}{x}$,



$$y = 1 + \cfrac{1}{1 + \cfrac{1}{x}},$$

and $y = x$. These two hyperbolas and the straight line, all intersect at the two points (1.618, 1.618) and ($^-$.618, $^-$.618) which are the two solutions of the equation $x^2 - x - 1 = 0$ and are ϕ and ϕ' (see chapter 7). This graphing of the "pieces" of the infinite continued fraction again was very exciting!

The infinite continued fraction for ϕ is

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}$$

One of the most interesting ideas came up when I was working with Sean, 8 years old at the time. I made up this equation off the top of my head. He solved it like this:

$$6x + 5 = 2x + 25 \qquad \text{add } {}^{-}5$$

$$6x = 2x + 20 \qquad \div 6 \text{ (unexpected)}$$

$$x = \frac{2x + 20}{6} \text{ and Eq. 8 } x = \frac{1}{3}x + 3\frac{1}{3}$$

He then said since $x = \frac{1}{3}x + \frac{2}{3}x$, then $\frac{2}{3}x = 3\frac{1}{3}$ and multiplying both sides by $\frac{3}{2}$, he got $x = 5$. Terrific. I then looked at Eq.8 and saw the x's on both sides and thought, hmmmm, could we do this like the quadratics above. I encouraged Sean to put the right side of Eq. 8 in for x on the right side.

And sure enough we ended up with an infinite series:

$$x = \left(\frac{1}{3}\right)^n \cdot x + 3\frac{1}{3} \cdot \left[\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \ldots + \left(\frac{1}{3}\right)^{n-1}\right]$$

as n approaches infinity $\left(\frac{1}{3}\right)^n$ goes to 0; Sean said "almost 0". Then

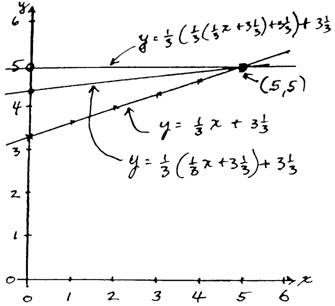
$x = 3\frac{1}{3} \cdot \left[1 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\right]$. From solving equation 8 the other way Sean knew that $x = 3\frac{1}{3} \cdot \frac{3}{2} = 5$

so $\left[1 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\right] = \frac{3}{2} = 1\frac{1}{2}$

and $\left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots = \frac{1}{2}$

So we arrived at the sum of an infinite series from solving a linear equation! A program for Eq.8 on the FX7000G would be:? →X◢ Lbl 7:$\frac{X}{3} + \frac{10}{3}$ → X◢ Goto 7. Put in 17 for X and we get the following infinite sequence which seems to converge to 5 (not surprising): 17, 9, 6.333 . . . , 5.444 . . . , 5.148 . . . , 5.049 . . . , 5.016 . . . , 5.005 . . .

Will this method work on all linear equations?



If we now graph $y = \frac{1}{3}x + 3\frac{1}{3}$

$y = \frac{1}{3}\left(\frac{1}{3}x + 3\frac{1}{3}\right) + 3\frac{1}{3}$ and

$y = \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}x + 3\frac{1}{3}\right) + 3\frac{1}{3}\right) + 3\frac{1}{3}$

we get 3 straight lines that intersect at—yes, (5,5)! and 5 is the solution to $x = \frac{1}{3}x + 3\frac{1}{3}$. So what happened with the quadratic equations also works for linear equations.

An infinite continued fraction for $\frac{4}{\pi}$, by Lord Brouncker, about 1658– see Olds:

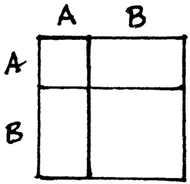
$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{\ddots}}}}}$$

An infinite continued fraction for e (see chapter 11), by Euler, 1737– see Olds:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\ddots}}}}}$$

CHAPTER 9: The Binomial Expansion and Infinite Series

Sean was 8 years-old when I asked him to do $(A + B)^2$. I showed him a square and asked him to find the area of each piece.



He came up with $A \cdot A + A \cdot B + B \cdot A + B \cdot B$ or $A^2 + 2 \cdot A \cdot B + B^2$

When he started on $(A + B)^3$ he very quickly decided that it would just be all the ways he could put 2 different things together using 3 at a time. We have a 3-D model he could also look at. He wrote:

$$(A + B)^3 = A \cdot A \cdot A + A \cdot A \cdot B + A \cdot B \cdot A + B \cdot A \cdot A + A \cdot B \cdot B + B \cdot A \cdot B + B \cdot B \cdot A + B \cdot B \cdot B$$

or $A^3 + 3 \cdot A^2B + 3 \cdot A \cdot B^2 + B^3$. From his method he could go on from there without any problem.

$$(A + B)^0 = 1$$
$$(A + B)^1 = A + B$$
$$(A + B)^2 = A^2 + 2 \cdot A \cdot B + B^2$$
$$(A + B)^3 = A^3 + 3 \cdot A^2 \cdot B + 3 \cdot A \cdot B^2 + B^3$$
$$(A + B)^4 = A^4 + 4 \cdot A^3 \cdot B + 6 \cdot A^2 \cdot B^2 + 4 \cdot A \cdot B^3 + B^4$$

With others, we talked about the distributive property to multiply two binomials, then multiplying a binomial by a trinomial to get the same results as above.

Then we try to find some patterns. What do you see about the exponents?

We'll write just the coefficients in this form:

Row #	Column #						
	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Now the question again, can you find some patterns in this triangle of numbers. This is sometimes called Pascal's Triangle (after Blaise Pascal 1632–1662, but known by Indian and Chinese thinkers 2000 years before) or Sean's or Ian's triangle). There are all kinds of patterns. STOP HERE TO TAKE A LOOK.

For about 35 years my students and I looked at these numbers and we found patterns in them, but what Ian did was something special (the very same Ian from Chapter 3). Ian did this when he was 12 years-old. Newton did a similar thing when he was 19, according to W. W. Sawyer in "Integrated Mathematics Scheme-Book C". Ian's words:

"I was faced with the problem of generating Pascal's triangle. I decided to start looking at patterns until I found one that applied to the entire triangle. After some trial and error, I noticed a pattern in the ratios from one column to the next. In row 4, for example, the ratios are arrived at by asking, what times 1 $= 4$? $\frac{4}{1}$. What times 4 $= 6$? $\frac{6}{4} = \frac{3}{2}$. Then $\frac{4}{6} = \frac{2}{3}$ and $\frac{1}{4}$. I wrote these ratios

$$\frac{4}{1} \quad \frac{3}{2} \quad \frac{2}{3} \quad \frac{1}{4}$$

$$\frac{4-0}{1} \quad \frac{4-1}{2} \quad \frac{4-2}{3} \quad \frac{4-3}{4}.$$

So to get the third number in the 4th row, I multiply

$$\frac{4-0}{1} \cdot \frac{4-1}{2} = \frac{4}{1} \cdot \frac{3}{2} = 6.$$

To get the 8th number in the 20th row:

$$\frac{20-0}{1}\cdot\frac{20-1}{2}\cdot\frac{20-2}{3}\cdot\frac{20-3}{4}\cdot\frac{20-4}{5}\cdot\frac{20-5}{6}\cdot\frac{20-6}{7} = 77520$$

The number in the x-th column, y-th row $= \dfrac{\dfrac{y!}{(y-x)!}}{x!}$."

[N. B. Ian changed from number in the row to column number. Since the columns start with the 0-th column, his *x* is one less than the number in the row, for example the 3rd number is in the 2nd column]. Ian's notation is the same as used to write the coefficients in combination form, to which Sean alluded to. Ian also introduced the factorial notation ($3! = 1 \cdot 2 \cdot 3 = 6$).

At this time Ian also worked on extending the triangle to the left with negative numbers. A couple of years earlier when he was in about 4th grade, Ian was graphing factorials, when he decided since these numbers got so big so fast, he graphed the log of the factorial. I wasn't even aware he knew about logs.

To write the general term of the binomial expansion, $(A + B)^n$, we will need Ian's number for each coefficient and the exponents for A and B. The exponent of B is the same as x, and since the exponents add up to n, the exponent of A is $n - x$. The general term in the x-th column and the n-th row is:

$$\frac{n!}{(n - x)!x!} A^{n-x}B^x$$

The first five terms of the binomial expansion calculated from the general term with x going from 0 to 4, would look like this:

$$(A + B)^n = A^n + nA^{n-1}B + \frac{n(n-1)}{1 \cdot 2}A^{n-2}B^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}A^{n-3}B^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}A^{n-4}B^4 + \ldots$$

One thing that made this work so worthwhile, was the connection I saw between the infinite series and the binomial expansion.

In Chapter 3, Ian came up with the sum of the infinite series:

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

(as Sean did also, but from his work in chapter 8). From chapter 1 came this infinite series

$$\frac{A}{B - A} = \frac{A}{B} + \left(\frac{A}{B}\right)^2 + \left(\frac{A}{B}\right)^3 + \ldots$$

If we put $A = Bx$ or $\frac{A}{B} = x$ in the equation above, we get

$$\frac{Bx}{B - Bx} = x + x^2 + x^3 + \ldots$$ reducing the left side

$$\frac{x}{1 - x} = x + x^2 + x^3 + \ ...$$

adding 1 to both sides—on the left side the $1 = \frac{1 - x}{1 - x}$, gives

$$\frac{x}{1 - x} + \frac{1 - x}{1 - x} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ ...$$

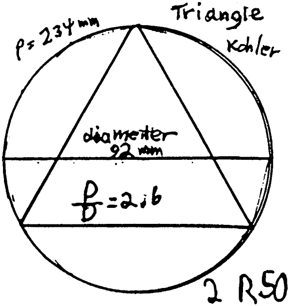
which is the infinite series Ian came up. Since

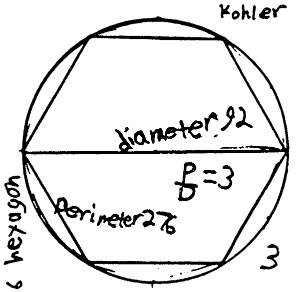
$$\frac{1}{1 - x} = (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ ...$$

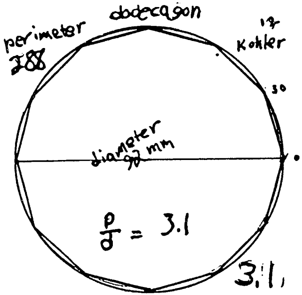
This is the important connection between the binomial expansion and the infinite series, because $(1 - x)^{-1}$ can be expanded by putting $1 \rightarrow A$, $^{-}x \rightarrow B$ and $^{-}1 \rightarrow n$ in $(A + B)^n$

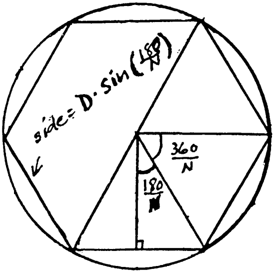
CHAPTER 10: π and Square Roots

1. Kholer, a 5th grader at the time, worked on drawing inscribed polygons in a 12-dot circle. He found the perimeter of the polygons, then divided this by the diameter of the circle. His numbers (only 3) formed a sequence, which got closer to π. This was similar to what Archimedes did.









If N = # of sides of the polygon (3, 6, 12, 24, . . .) and F goes from $0 \rightarrow \infty$, a program for the Casio FX7000G to get the ratio $\frac{P}{D} = x$, that Kholer was doing is:

$0 \rightarrow F : \text{Lbl } 4 : 3 \cdot 2^F \rightarrow N : N \cdot \sin\left(\frac{180}{N}\right) \rightarrow X$◢ ISZ F: Goto 4. The infinite sequence we get is 2.59807, 3, 3.1058, 3.1326, 3.1393, . . . and after 10 loops we get 3.14159 . . . correct to 5 decimal places for π. Notice the first 3 are very much what Kholer had.

2. Sean was reading "The History of Pi" and found the Gregory-Leibnitz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Sean wrote a program to do this, and found the series converges slowly.

3. Wells: "Tamura and Kanada calculate π to 16 million places, based on Gauss' study of arithmetic-geometric mean of two numbers. The initial values are, $A = X = 1$, $B = \frac{1}{\sqrt{2}}$ and $C = \frac{1}{4}$. The program steps follow:

$$Y = A;\ A = \frac{A + B}{2};\ B = \sqrt{B \cdot Y};\ C = C - X \cdot (A - Y)^2;$$

$$X = 2 \cdot X;\ \text{PRINT } \frac{(A + B)^2}{4 \cdot C};\ \text{go back to first step.}$$

It has the amazing property that the number of correct digits approximately doubles with each circuit of the loop".

4. One day one of my students, in a moment of trying to do nothing, started to continuously hit the $\sqrt{\ \ }$ of a number on the calculator. He took $\sqrt{5}$, then $\sqrt{\ \ }$ of the first answer, etc. Lo and behold he came up with an infinite sequence of numbers whose limit is 1. Try it! Will this work for any number? Try .5. Briggs in 1620 used successive square roots of 10 to calculate logarithms, so this turns out to be important.

5. To find $\sqrt{2}$ by squaring numbers to nearest whole number, tenth, hundredth, . . . We get two infinite sequences, one made up of the smallest numbers that are too big, the other made up of the biggest numbers that are too small.

Too big: 2, 1.5, 1.42, 1.415, 1.4143, 1.41422, 1.414214, . . .

Too small: 1, 1.4, 1.41, 1.414, 1.4142, 1.41420, 1.414213, . . .

Both sequences approach $\sqrt{2}$ as a limit.

6. Find the $\sqrt{40}$ by averaging: We're trying to find two numbers the same, which when multiplied give 40. Suppose we guess 5. If 5 is too small, then $\frac{40}{5} = 8$ is too big. And the $\sqrt{40}$ must lie between 5 and 8. So we take the average of 5 and $\frac{40}{5}$ and use that as the new guess number and continue to do this obtaining an infinite sequence approaching $\sqrt{40}$. To generalize, if G is our guess number, N is the number whose square root we are trying to find, then Gayla's (a 6th-grader) program to find the first 20 approximations for $\sqrt{N}$ would look like this:

```
10 C = 0
20 INPUT N
30 INPUT G
40 PRINT G
```

$$50\ G = \frac{\frac{N}{G} + G}{2}$$

```
60 C = C + 1
70 PRINT G
80 IF C < 19 THEN GOTO 50
90 STOP
```

(The approximatations to $\sqrt{40}$ are 5, 6.5, 6.326924, 6.324556, . . . converging very quickly).

7. Find $\sqrt{2}$ using the binomial expansion:

Ian's proof earlier (chapter 3), that infinity $= {}^{-}1$ shows that in $(1 - x)^n$, if n is negative or a fraction, x has to be less than 1, otherwise strange things happen.

$\sqrt{2} = 2^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{-1}{2}} = \left(1 - \frac{1}{2}\right)^{\frac{-1}{2}}$ now it's in the binomial form, $(A + B)^n$; if we

put 1 into A, $\frac{-1}{2}$ in for B and $\frac{-1}{2}$ in for n.

$$\sqrt{2} = \left(1 - \frac{1}{2}\right)^{\frac{-1}{2}} = 1^{\frac{-1}{2}} + \left(\frac{-1}{2}\right) \cdot 1^{\frac{-1}{2} - 1} \cdot \left(\frac{-1}{2}\right)^1 +$$

$$\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2} - 1\right)}{2!} \cdot 1^{\frac{-1}{2} - 2} \cdot \left(\frac{-1}{2}\right)^2 +$$

$$\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2} - 1\right)\left(\frac{-1}{2} - 2\right)}{3!} \cdot 1^{\frac{-1}{2} - 3} \cdot \left(\frac{-1}{2}\right)^3 + \ldots$$

$$\sqrt{2} = 1 + .25 + .09375 + .0390625 + .017089844 + .007690430 + \ldots$$

$\sqrt{2} = 1.407592774 \ldots$ using 6 terms as an approximation of this infinite series.

CHAPTER 11: Compound Interest to e

At one point, when Ian was 11 years old, he came in asking how much would his father have to pay in monthly installments on a house worth \$10,000 at 10% interest over 30 years. This was a problem similar to one which one of my parents asked me to figure out for his accounting firm a while before. This got us working on this and the simpler problem of finding the interest if it compounded, that is, added to what you put in the bank.

Let's look at the problem of finding *simple interest* first. Suppose you put \$1 in the bank at a 7% annual rate of interest. How much would you have after 1 year?

The Simple Interest = Principal · rate · time(in years)

Int. $= \$1 \cdot .07 \cdot 1 = \$.07$ for 1 year. So you would have \$1.07 after one year. After 2 years you would get interest of $I = 1 \cdot .07 \cdot 2 = \$.14$. After 2 years then, you would have \$1.14. After 3 years you would have \$1.21.

So the interest is not added on when you figure the next interest, but always figured on $1.

Let's figure out what you would have in the bank after 3 years, putting in $1, at 7% interest, if the interest is *compounded annually*:

Int. (earned first year) $= \$1 \cdot .07 \cdot 1 = \$.07$ (same as simple interest)
Amount you have after 1 year $= 1 + .07 = \$1.07$, but we'll leave it in the form $1 + .07$, also the same as simple interest.

Int. (earned 2nd yr) $= (1 + .07) \cdot .07 \cdot 1 = (1 + .07) \cdot .07 = .0749$ which is more than for the simple interest case.

Am't (after 2nd yr) = Am't. end of 1st yr. + Int. during 2nd yr.
Am't (after 2nd yr) $= (1 + .07) + (1 + .07) \cdot .07$
Am't (after 2nd yr) $= 1 + .07 + .07 + .07^2$
Am't (after 2nd yr) $= (1 + .07)^2 = 1.1449$, just a little more than the simple interest case.

Notice also, we have a binomial expansion problem $(A + B)^2$.

Let's go on one more year.
Int. (3rd yr) $= (1 + .07)^2 \cdot .07 \cdot 1$
Am't (3 yrs) = Am't. after 2 years + Int. during 3rd yr.
(line 1) Am't(3 yrs) $= (1 + .07)^2 + (1 + .07)^2 \cdot .07 =$
Am't (3 yrs) $= 1 + 2 \cdot .07 + .07^2 + .07 + 2 \cdot .07^2 + .07^3 =$
Am't (3 yrs) $= 1 + 3 \cdot .07 + 3 \cdot .07^2 + .07^3 = (1 + .07)^3 = \$1.225$

In line 1, just factor out $(1 + .07)^2$ to get $(1 + .07)^3 = \$1.225$ which is more than the \$1.21 in simple interest. And after 10 years the amount you would have is $(1 + .07)^{10} = 1.96$. which means at 7% interest compounded annually you would double your money in a little over 10 years. If you put \$3 in the bank at 7% for 10 years, compounded annually, you would have $3 \cdot (1 + .07)^{10} = 3 \cdot \$1.96$. In general, if A = the amount you have in the bank after t years, and X = the annual rate of interest, *compounded annually,* then

$$A = P \cdot (1 + X)^t$$

I set the next sequence of problems to Sean, at age 9, and a couple of teachers at a UI workshop:

What if we compound the interest semi-annually or twice a year? Would that make a big difference in the amount we have after 1 year? Let's try the same annual rate of interest 7%, on $1 again.

$$\text{Int. (first } \tfrac{1}{2} \text{ yr)} = \$1 \cdot .07 \cdot \frac{1}{2} = \frac{.07}{2} \; (3\tfrac{1}{2} \text{ cents})$$

$$\text{A (end of } \tfrac{1}{2} \text{ yr)} = 1 + \frac{.07}{2}$$

$$\text{Int. (second } \tfrac{1}{2} \text{ yr)} = \left(1 + \frac{.07}{2}\right) \cdot .07 \cdot \frac{1}{2}$$

$$\text{Am't (after } \tfrac{2}{2} \text{ yr or 1 yr)} = \left(1 + \frac{.07}{2}\right) + \left(1 + \frac{.07}{2}\right) \cdot \frac{.07}{2}$$

$$\text{Am't (after } \tfrac{2}{2} \text{ yr or 1 yr)} = \left(1 + \frac{.07}{2}\right)^2 = 1.071225,$$ which is a little more than when compounded annually.

What if we compounded the interest monthly (12 times per year), same $1 at the same 7% for 1 year? It would just be

$$\text{Am't (after } \tfrac{12}{12} \text{ yr or 1 yr)} = \left(1 + \frac{.07}{12}\right)^{12} = 1.072290081$$

Compounded daily (365 times per year)?

$$\text{Am't (after } \frac{365}{365} \text{ yr or 1 yr)} = \left(1 + \frac{.07}{365}\right)^{365} = 1.072500983$$

Compounded 1 million times per year?

$$\text{Am't (after 1 yr)} = \left(1 + \frac{.07}{1\text{M}}\right)^{1\text{M}} = 1.072508179 \ldots$$

This looks like an infinite sequence

1,	2,	. .	12,	. .	365,	. .	1,000,000	. . .
1.07,	1.071225,	. . ,	1.072290081,	. . ,	1.072500983,	. .	1.072508179	. . .

So . . . no matter how many times the interest is compounded per year, the amount you have after 1 year, at a 7% annual interest rate, will never be bigger than \$1.08 to the nearest cent!

This number $\left(1 + \frac{.07}{n}\right)^{n}$ as n goes to infinity equals $e^{.07} = 1.072508181 \ldots$ and we could write the rule $A = P \cdot e^{xt}$, the amount you would have at the end of t years, at x annual rate of interest, compounded continuously.

If we expand $\left(1 + \frac{.07}{n}\right)^n$ using the binomial expansion we get the infinite series:

$$\left(1 + \frac{.07}{n}\right)^n = 1^n + n \cdot 1^{n-1} \cdot \frac{.07}{n} + \frac{n(n-1)}{2!} \cdot 1^{n-2} \cdot \left(\frac{.07}{n}\right)^2 +$$

$$\frac{n(n-1)(n-2)}{3!} \cdot 1^{n-3} \cdot \left(\frac{.07}{n}\right)^3 + \ldots$$

Simplifying, 1 to any power is 1 and just switching the factorials and powers of n

$$\left(1 + \frac{.07}{n}\right)^n = 1 + .07 + \frac{n(n-1)}{n^2} \cdot \frac{.07^2}{2!} +$$

$$\frac{n(n-1)(n-2)}{n^3} \cdot \frac{.07^3}{3!} + \frac{n(n-1)(n-2)(n-3)}{n^4} \cdot \frac{.07^4}{4!} + \ldots$$

Since all the coefficients with n in them go to 1 as $n \to \infty$, the limit of $\left(1 + \frac{.07}{n}\right)^n$, as $n \rightarrow \infty$, $= 1 + .07 + \frac{.07^2}{2!} + \frac{.07^3}{3!} + \frac{.07^4}{4!} + \frac{.07^5}{5!} +$ $\ldots = e^{.07} = 1.072508181 \ldots$

The limit of $\left(1 + \frac{1}{n}\right)^n$ as $n \rightarrow \infty = e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots =$ $2.718281828 \ldots$(not rational, but transcendental, like π); Ian had this figured out at age 12.

And the limit of $\left(1 + \frac{x}{n}\right)^n$, as $n \rightarrow \infty$, $= e^x$, and as an infinite series generalizing from above, by Sean and the teachers in 1987 and by Newton in 1669:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

e is a very important number in mathematics; not only does it show up in this compound interest problem, but as the base of the natural logarithms as well as in exponential growth and decay and can be written in terms of sines and cosines.

I'd like to take a jump here, because what follows really was one of the highlights of my college math. I was so excited about this result, I painted a picture showing it. Five of the most important numbers in mathematics come together in one true statement! This will give you something to think about for a long while, as it did me.

In the following, i represents $\sqrt{^-1}$, the imaginary number such that $i \cdot i = i^2 = {}^-1$, and $i = i$, then $i^3 = {}^-i$, $i^4 = 1$, $i^5 = i$, $i^6 = {}^-1$, etc. In place of x in the series for e^x above we'll put ia, a being a number in radians, and we get

$$e^{ia} = 1 + ia + \frac{(ia)^2}{2!} + \frac{(ia)^3}{3!} + \frac{(ia)^4}{4!} + \frac{(ia)^5}{5!} + \ldots$$

$$e^{ia} = 1 + ia - \frac{a^2}{2!} - \frac{ia^3}{3!} + \frac{a^4}{4!} + \frac{ia^5}{5!} - \frac{a^6}{6!} - \ldots$$

Separating the real parts from the imaginary parts,

$$e^{ia} = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \ldots + i \cdot \left(a - \frac{a^3}{3!} + \frac{a^5}{5!} - \ldots\right)$$

According to Griffiths, Newton did not show how he figured out the infinite series for cosine and sine, but he did this:

$$\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \ldots \text{ and}$$

$$\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \ldots$$

It was Cotes by 1716 who discovered that:

$$e^{ia} = \cos a + i \cdot \sin a$$

If we now put π radians in for a above, we get $e^{i\pi} = \cos\pi + i \cdot \sin\pi$. Since $\cos\pi = {}^{-}1$ and $\sin\pi = 0$, we arrive at $e^{i\pi} = {}^{-}1 + 0$. Adding 1 to both sides we get the amazing result

$$\boxed{e^{i\pi} + 1 = 0}$$

This is the true statement that has the five most important numbers in mathematics in it: e, i, π, 1 and 0 . Wow!

"In his first paper on the Calculus (1669), Newton proudly introduced the use of infinite series to expedite the processes of the calculus . . .

As Newton, Leibnitz, the several Bernoullis, Euler, d'Alembert, Lagrange, and other 18th-century men struggled with the strange problem of infinite series and employed them in analysis, they perpetuated all sorts of blunders, made false proofs, and drew incorrect conclusions; they even gave arguments that now with hindsight we are obliged to call ludicrous."

CHAPTER 12: The Two Problems of the Calculus

1. *The derivative,* or "rate of change", is used to describe how quickly quantities change. Historically, it helped deal with problems of how a pendulum swings, how the planets move in their orbits, the velocity of a cannon ball, light, and electrons in a wire; and 2. *the integral,* which turns out to be just the inverse of the derivative, historically dealt with finding the area under curves, the volume of fairly regular shapes, the work done in moving objects and the energy in light scattering.

Both problems involve infinite sequences and their limits. The derivative deals with an infinite sequence of slopes of lines, the integral deals with an infinite sequence of areas under a curve (which in turn involves an infinite series).

Archimedes (287–212 B. C.) essentially invented the integral calculus. We'll start there and do it in different ways.

CHAPTER 13: Area Under Curves—The Integral

1. ARCHIMEDES found the area within a parabolic segment shown in Fig. 1—the area enclosed by the parabola $y = x^2$ and the horizontal line segment:

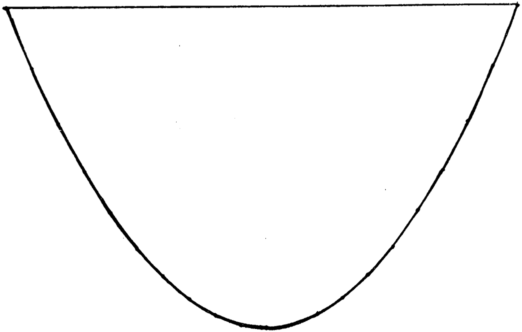


Fig. 1

Figure 2 shows a triangle of the same base and height as the parabolic segment. Archimedes proved that the area of the parabolic segment is $\frac{4}{3}$ the area of this triangle, say T.

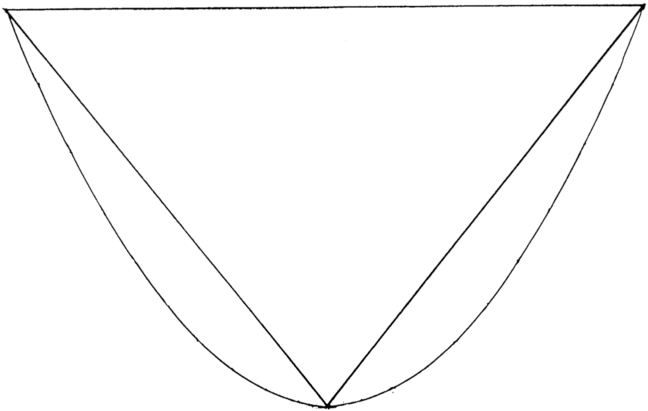


Fig. 2

Figure 3. Archimedes built two smaller triangles between the big triangle and the parabola by constructing a perpendicular to *AJ* at its mid-point *H*, intersecting the parabola at a point, call it *F*. He then showed that the area of the 2 smaller triangles combined, is $\frac{1}{4}$ of T.

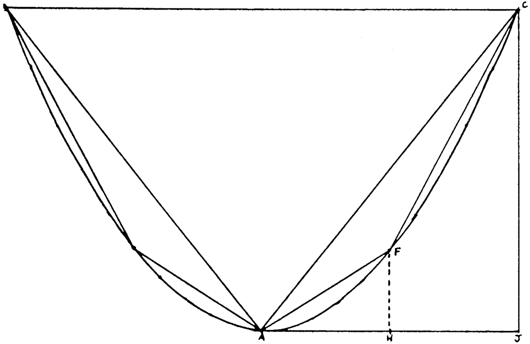


Fig. 3

Figure 4 shows how he then builds 4 smaller triangles, whose combined area is $\frac{1}{4}$ of $\frac{1}{4}$ of T.

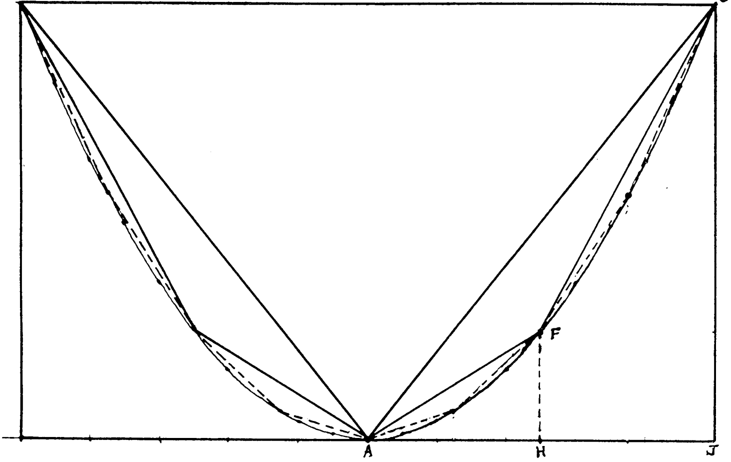


Fig. 4

Archimedes continues this process.

The area of the parabolic segment =

$$= T + \left(\frac{1}{4}\right)T + \left(\frac{1}{4}\right)^2 T + \left(\frac{1}{4}\right)^3 T + \dots$$

$$= T + T \cdot \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots\right].$$

From chapter 1, the infinite series in the brackets converges, with limit $\frac{1}{3}$.

So the area of the parabolic segment $= T + \frac{1}{3} \cdot T = \frac{4}{3} \cdot T$.

At the time of Archimedes they didn't accept the idea of infinite nor did he use the idea of limit, but instead showed that the area couldn't be bigger than $\frac{4}{3} \cdot T$ and it couldn't be smaller than $\frac{4}{3} \cdot T$ so it had to equal $\frac{4}{3} \cdot T$.

Using figure 5, I show my version of why the area of the 2 smaller triangles = $\frac{1}{4} \cdot T$ by showing that the area of triangle *ACF* is $\frac{1}{4}$ the area of triangle *ACB*. The proof uses the idea that if two triangles have the same base, their areas are proportional to their altitudes. From figure 3., *HF* is extended up to intersect *AC*, at a point we call *E*. *AC* is a base of triangle *ABC* and triangle *ACF*. *BG* and *FD* are their corresponding altitudes. Triangle *DEF* is similar to triangle *GAB* because they each have two angles equal, the dot angle (because *EFH* is parallel to *AB*) and the right angle.

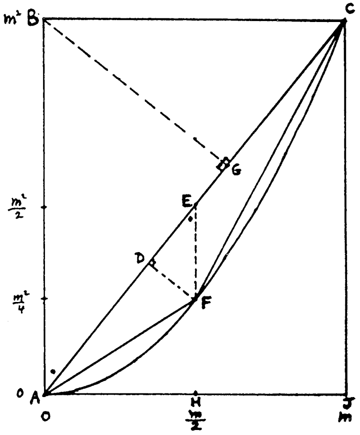


Figure 5

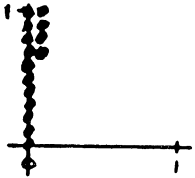
$EF = \frac{1}{4} \cdot AB$ because EF is $\frac{1}{2}$ of EH which is $\frac{1}{2}$ of AB. Then $DF = \frac{1}{4} \cdot GB$ and the area of triangle $ACF = \frac{1}{4}$ the area of triangle ACB.

The integral is just the area *under* the curve, which we'll find now. Archimedes showed that the area of the parabolic segment is $\frac{4}{3} \cdot$ T. In figure 5 above then, the area of $\frac{1}{2}$ the parabolic segment $= \frac{1}{2} \cdot \frac{4}{3} \cdot \mathrm{T} = \frac{2}{3} \cdot$ T. T is also the area of the rectangle $AJCB$. Therefore, the area below the parabola $= \frac{1}{3} \cdot$ T. If AJ goes from 0 to m with length m, then JC goes from 0 to m^2 and has length m^2. Another name for the area of the rectangle $AJCB$ is $m \cdot m^2 = m^3$.

So the area under the parabola $y = x^2$ from 0 to $m = \frac{1}{3} \cdot m^3$. Another way to say this is that the integral of x^2 from 0 to m, is $\frac{1}{3} \cdot m^3$.

2. PLOTTING POINTS (USING THE CASIO FX7000G), TO FIND THE AREA UNDER CURVES:

I was browsing through a Scientific American "Computer Recreations" article in which it looked like they were filling in squares on a computer screen. I asked myself "Could I do that on our Casio FX7000G programmable graphics calculator?". After a couple of hours of trying things and making mistakes (I was never good at programming), I was able to plot points on the screen to make a square. I plotted the point (0,1), subtracted a little from the y-coordinate then plotted another point, until the y-coordinate went to zero. Then I added a little to the x-coordinate, and again plotted points from y = 1 to zero and so on until x = 1, to make a 1 by 1 square.



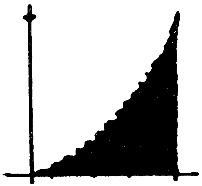
The next question I asked myself was "Could I find the area of the figure?". If I counted the points, that would give me a measure of the area. The program ended up looking like this (there are no line numbers needed but they are numbered for reference later):

1)	Range $-2.35, 2.35, 1, -1.55, 1.55, 1$	
		(The range puts in my x min, x max, x scale, y min, y max, y scale. The 2.35 to 1.55 ratio is used to make a square grid on a rectangular screen of a 95×63-dot display)
2)	$0 \rightarrow x$	(sets the left side of the figure).
3)	$0 \rightarrow N$	(N keeps count of the number of points plotted)
4)	Lbl 3	(place to which Goto 3 returns)
5)	$1 \rightarrow y$	(sets the top of the figure)
6)	Lbl 4	(place to which Goto 4 returns)
7)	Plot x,y	(plots the point)
8)	$1 + N \rightarrow N$	(adds 1 to the counter)

9)	$y - .07 \rightarrow y$	(I had to play around to get .07. If this number is too big all the spaces would not be filled in; if it was too small the number of points plotted wouldn't be a minimum.)
10)	$y > 0 \Rightarrow$ Goto 4	(Sets the bottom of the figure. If $y > 0$ it jumps to Lbl 4, otherwise goes to step 11)
11)	$x + .07 \rightarrow x$	(moves the plotting to the right)
12)	$x < 1 \Rightarrow$ Goto 3	(sets the right end of the figure)
13)	"N = " N◢	(Displays the number of points; hitting the G $\leftrightarrow$ T key shows the picture)

Running the program above fills in a 1×1 square; the number of dots is 400. If a 2 is used instead of 1 in line 12, I would get a 1×2 rectangle. The number of dots is 800, so the area of the 1×2 rectangle is $\frac{800}{400} = 2$.

The next question I asked was "Could I find the area under the curve $y = x^2$ from 0 to 1?". Easy. Just replace the 1 in line 5 with x^2, that's it! We get the picture below:



and $N = 133$.

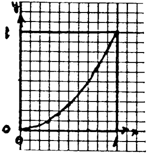
So the area under the curve $y = x^2$ from 0 to 1 is $\frac{133}{400}$ which is very close to $\frac{1}{3}$, which is the integral of x^2, from 0 to 1. I was excited about this and showed it to many people I work with, ages 6 to 45, as well as teachers, principals and parents.

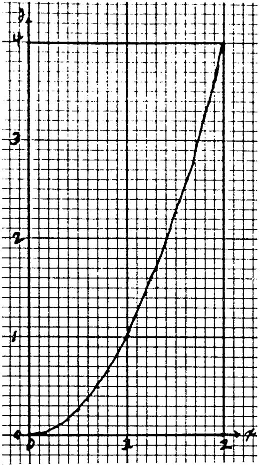
When I asked a couple of 6th graders what they would expect by just looking at the picture above, one said less than $\frac{1}{2}$ the 1×1 square, the other said $\frac{1}{3}$ of it. I was able to get the area under $y = x^3$, within a circle, an ellipse, between curves and under the normal distribution curve.

Once I saw how simple this was, it gave me the impetus to get young people thinking about the integral.

Now I asked "How could I do this without the calculator?"

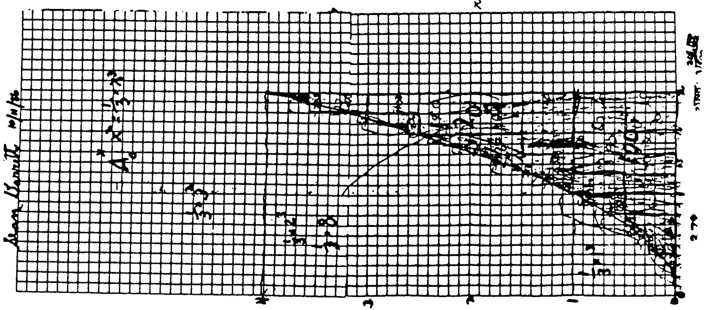
Sean, a gifted 8 year-old, guessed correctly from the picture on the calculator, about the area under $y = x^2$ from 0 to 1 and the area from 0 to 2 would be $\frac{1}{3} \cdot 8$. Then upon my suggestion, he proceeded to graph $y = x^2$ on graph paper and actually counted the squares under the curve. We ended up using $\frac{1}{10}''$ graph paper for this work on integrals.





Matt, a 7-year-old, counted 32 squares, and $\frac{32}{100}$ of the 1×1 square as the area under $y = x^2$ from $x = 0$ to $x = 1$. I asked what simple fraction $\frac{33}{100}$ equals. To which he replied, $\frac{1}{3}$. From 0 to 2 he correctly predicted $\frac{1}{3}$ of the 2×2^2 rectangle or $\frac{1}{3} \cdot 2^3$ as the area under $y = x^2$ from $x = 0$ to $x = 2$.

Sean counted an approximate 279 squares in going from 0 to 2, but was satisfied this was close enough (see his figure below). He figured there are $20 \cdot 40 = 800$ squares in the 2×4 rectangle ($2 \cdot 4 = 8$ square units), and the area should be $\frac{1}{3}$ of 800 or 266.666 . . . which is close to his 279. I don't like the normal integral notation for this work with Sean and other young people. I used this: $A_{0\text{ to }1}(x^2) = \frac{1}{3} = \frac{1}{3} \cdot 1^3$. This means, the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ is $\frac{1}{3}$, and we later used $\frac{1}{3}$ of 1^3 because he saw the pattern. Then $A_{0\text{ to }2}(x^2) = \frac{1}{3} \cdot 8 = \frac{1}{3} \cdot 2^3$ and $A_{0\text{ to }3}(x^2) = \frac{1}{3} \cdot 27 = \frac{1}{3} \cdot 3^3$ and he wrote $A_{0\text{ to }x}(x^2) = \frac{1}{3} \cdot x^3$. After he had done a similar thing for $y = x^3$, $A_{0\text{ to }x}(x^3) = \frac{1}{4} \cdot x^4$ he generalized to $A_{0\text{ to }x}(x^n) = \frac{x^{n+1}}{n+1}$. Very exciting!



As a teacher, I need to try new things, do mathematics and look for ways to get my students into more difficult concepts, but at their level. I treat each student as an individual; I do different things with different students. And I don't wait until I completely understand everything about an idea before I'll get a student doing it—that way I learn new things along with my students. It's also why I encourage them to do things different ways. It makes teaching enjoyable.

Much of the above section on plotting points under the curve was printed in the September 1987 issue of The Illinois Mathematics Teacher.

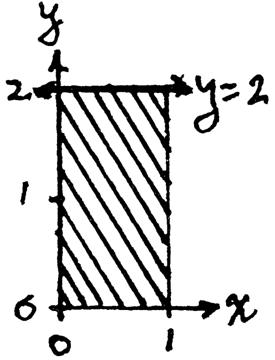
4. THE RECTANGLE METHOD (THE STANDARD TEXTBOOK METHOD) OF FINDING THE AREA UNDER CURVES

We'll begin with some problems which you can solve simply, like finding the area under straight lines. We just have to find the area of rectangles and triangles, without infinite series here.

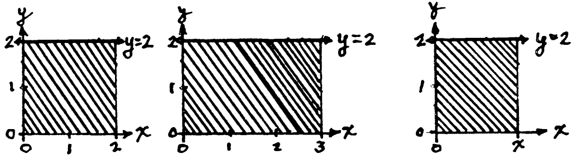
The first problem is to find the area under the curve $y = 2$ from $x = 0$ to $x = 1$

The area is just the area of a rectangle, $2 \cdot 1$. We will use this notation to show it:

$$A_{0 \text{ to } 1}(2) = 2 \cdot 1$$



The next problem: To find the area under the curve $y = 2$ from $x = 0$ to $x = 2$, 3, and x.

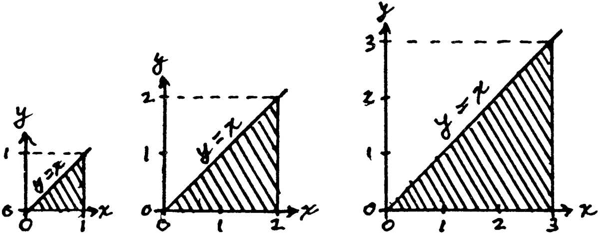


The area under $y = 2$ from $x = 0$ to $x = 1$ is $A_{0 \text{ to } 1}(2) = 2 \cdot 1$
The area under $y = 2$ from $x = 0$ to $x = 2$ is $A_{0 \text{ to } 2}(2) = 2 \cdot 2$
The area under $y = 2$ from $x = 0$ to $x = x$ is $A_{0 \text{ to } x}(2) = 2 \cdot x$ and

The area under $y = 5$ from $x = 0$ to $x = x$ is $A_{0 \text{ to } x}(5) = 5 \cdot x$ and

The area under $y = a$ from $x = 0$ to $x = x$ is $A_{0 \text{ to } x}(a) = a \cdot x$

The NEW problem: find the area under the curve $y = x$ from $x = 0$ to $x = 1, 2$ and 3. It's a triangle, so predict the area first.



The area under $y = x$ from $x = 0$ to $x = 1$ is $A_{0\text{ to }1}(x) = \frac{1}{2} \cdot 1^2$

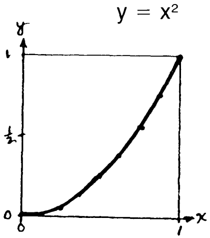
The area under $y = x$ from $x = 0$ to $x = 2$ is $A_{0\text{ to }2}(x) = \frac{1}{2} \cdot 2^2$

The area under $y = x$ from $x = 0$ to $x = 3$ is $A_{0\text{ to }3}(x) = \frac{1}{2} \cdot 3^2$

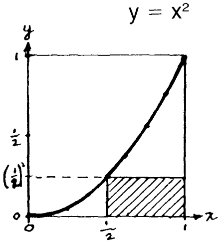
The area under $y = x$ from $x = 0$ to $x = x$ is $A_{0\text{ to }x}(x) = \frac{1}{2} \cdot x^2$

The area under $y = a \cdot x$ from $x = 0$ to $x = x$ is $A_{0\text{ to }x}(a \cdot x) = a \cdot \frac{1}{2} \cdot x^2$

The NEW problem: find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$. The strategy will be this, we'll cut the horizontal section (0 to 1, in this case) into 2 pieces, then 3 pieces, etc. We'll get rectangles to fit under the curve, find the area of each and add them up. In each case the width of the rectangles $\left(\frac{1}{n}\right)$ will get smaller and smaller and the number of them $(n - 1)$ will increase. So we will get an infinite sequence of areas $a_1, a_2, a_3, \ldots a_n$ which will approach the total area under the curve from 0 to whatever and in the limit, as $n \to \infty$, will be the total area or *integral*.

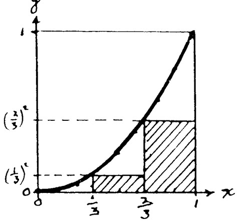


$a_1 = 0$

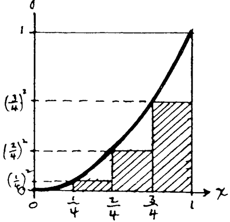


$a_2 = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}$

$y = x^2$



$y = x^2$



$$a_3 = \left(\frac{1}{3}\right)^2 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{1}{3^3} \cdot (1^2 + 2^2)$$

$$a_4 = \left(\frac{1}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{2}{4}\right)^2 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4^3} \cdot (1^2 + 2^2 + 3^2)$$

$$a_n = \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{3}{n}\right)^2 \cdot \frac{1}{n} + \ldots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n}$$

$$a_n = \frac{1}{n^3} \cdot [1^2 + 2^2 + 3^2 + \ldots + (n-1)^2]$$

In the brackets we have a function, the sum of the squares of the numbers from 1 to $n - 1$; 1, 5, 14, 30 . . . (see problem 5 in chapter 6). The sum of the squares from 1 to n is $\frac{1}{6} \cdot n \cdot (n + 1) \cdot (2n + 1)$. So the sum of the squares from 1 to $n - 1$ is $\frac{1}{6} \cdot n \cdot (n - 1) \cdot (2n - 1) = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$. Putting this in the brackets above $a_n = \frac{1}{n^3} \cdot \left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}\right)$ or

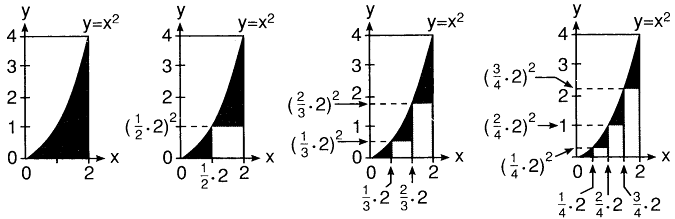
$$a_n = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

which is the nth term of the sequence $a_1, a_2, a_3, \ldots, a_n$, the sequence of areas under the curve.

The area under the curve $y = x^2$ from $x = 0$ to $x = 1$, is the limit of a_n, as $n \to \infty$, $= A_{0 \text{ to } 1}(x^2) = \frac{1}{3} = \frac{1}{3} \cdot 1^3$ (as $n \to \infty$, both terms $\frac{1}{2n}$ and $\frac{1}{6n^2} \to 0$)

The $\frac{1}{3}$ is the same as Sean, Matt and Archimedes got above. Writing the area as $\frac{1}{3} \cdot 1^3$ is important in that it shows the area is $\frac{1}{3}$ of the 1×1^2 rectangle (in this case square) around the curve and will help to see the pattern.

The NEXT problem: find the area under the curve $y=x^2$ from $x=0$ to $x=2$



TRY TO FIND the area of the shaded portions, a_1, a_2, a_3, a_4, and a_n BEFORE YOU GO ON.

$$a_1 = 0$$

$$a_2 = \left(\frac{1}{2} \cdot 2\right)^2 \cdot \frac{1}{2} \cdot 2 = \frac{2^3}{2^3} \cdot 1^2$$

$$a_3 = \left(\frac{1}{3} \cdot 2\right)^2 \cdot \frac{1}{3} \cdot 2 + \left(\frac{2}{3} \cdot 2\right)^2 \cdot \frac{1}{3} \cdot 2 = \frac{2^3}{3^3} \cdot (1^2 + 2^2)$$

$$a_4 = \left(\frac{1}{4} \cdot 2\right)^2 \cdot \frac{1}{4} \cdot 2 + \left(\frac{2}{4} \cdot 2\right)^2 \cdot \frac{2}{4} \cdot 2 + \left(\frac{3}{4} \cdot 2\right)^2 \cdot \frac{3}{4} \cdot 2$$

$$= \frac{2^3}{4^3} \cdot (1^2 + 2^2 + 3^2)$$

$$a_n = \left(\frac{1}{n} \cdot 2\right)^2 \cdot \frac{1}{n} \cdot 2 + \left(\frac{2}{n} \cdot 2\right)^2 \cdot \frac{1}{n} \cdot 2 + \left(\frac{3}{n} \cdot 2\right)^2 \cdot \frac{1}{n} \cdot 2 + \ldots$$

$$+ \left(\frac{n-1}{n} \cdot 2\right)^2 \cdot \frac{1}{n} \cdot 2 = \frac{2^3}{n^3} \cdot [1^2 + 2^2 + 3^2 + \ldots (n-1)^2]$$

$$a_n = \frac{2^3}{n^3} \cdot \left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}\right) \text{ or}$$

$$a_n = 2^3 \cdot \left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right)$$

The area under the curve $y = x^2$ from $x = 0$ to $x = 2$ is the limit of a_n as $n \to \infty$, $= A_{0 \text{ to } 2}(x^2) = \frac{1}{3} \cdot 2^3$ (as $n \to \infty$, $\frac{1}{2n} \to 0$ and $\frac{1}{6n^2} \to 0$).

Patterns now!

$A_{0 \text{ to } 1}(x^2) = \frac{1}{3} \cdot 1^3$

$A_{0 \text{ to } 2}(x^2) = \frac{1}{3} \cdot 2^3$ What do you guess for $A_{0 \text{ to } 3}(x^2)$? $A_{0 \text{ to } x}(x^2)$?

Look for patterns now in what we've done so far!

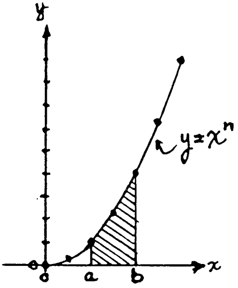
$A_{0 \text{ to } x}(1) = x$; $A_{0 \text{ to } x}(x) = \frac{1}{2} \cdot x^2$; $A_{0 \text{ to } x}(x^2) = \frac{1}{3} \cdot x^3$;

What about $A_{0 \text{ to } x}(x^3)$? And then $A_{0 \text{ to } x}(x^n) = ?$

As Sean figured out the other way, the area under the curve $y = x^n$ from $x = 0$ to $x = x$ is $A_{0 \text{ to } x}(x^n) = \frac{1}{n+1} \cdot x^{n+1}$.

Find the area under these curves, from $x = 0$ to x: $y = 2x^3 + 5x$; $y = 7x^2 - 3x + 5$.

Now, how about the area under the curve $y = x^n$ from $x = a$ to $x = b$? Here's the graph:



As you might expect, just subtract the area under the curve from 0 to a from the area under the curve from 0 to b; in other words,

$$A_{a \text{ to } b}(x^n) = \frac{1}{n+1} \cdot b^{n+1} - \frac{1}{n+1} \cdot a^{n+1}$$

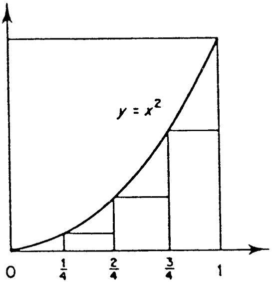
4. TWO PROBLEMS

Two seemingly unrelated problems ended up with the same solution.

1. The area under the parabola $y = x^2$, from $x = 0$ to $x = 1$, approaches $\frac{1}{3}$.

2. The ratio of the volume of a square pyramid to the volume of a cube with the same base, approaches $\frac{1}{3}$.

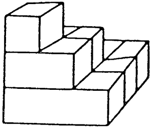
PROBLEM 1: If the area under the parabola is done by finding the sum of the areas of rectangles (an infinite series)

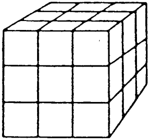


$$\begin{aligned} A &= \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{2}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 \\ &= \frac{1^2}{4^3} + \frac{2^2}{4^3} + \frac{3^2}{4^3} \\ &= \frac{1^2 + 2^2 + 3^2}{4^3} \end{aligned}$$

we come up with the sum of $(n - 1)$ squares divided by n^2, so the area turns out to be $\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$. As $n \to \infty$, the last two terms go to 0, and the area goes to $\frac{1}{3}$.

PROBLEM 2: If the pyramid and cube are made with Cuisenaire rods,





the volume of the pyramid is $1^2 + 2^2 + 3^2$; the volume of the cube is 3^3; the ratios form an infinite sequence, the 3rd of which is

$$\frac{\text{Vol. of pyramid}}{\text{Vol. of cube}} = \frac{1^2 + 2^2 + 3^2}{3^3}, \text{ generalizing to}$$

$\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$. As $n \to \infty$, this ratio approaches $\frac{1}{3}$ as a limit, as does the area under the parabola!

That summer Grace figured out PROBLEM 1 to get the rule and later her younger brother Richard worked on PROBLEM 2. He did not get the rule, but we programmed a calculator to find the ratio of the volume of the pyramid to the volume of the cube. We got the infinite sequence 1, .625, .518, .469, .440, .421, .408, .398, .390, .385, . . . and for $n = 200$ about .335, a slowly converging sequence.

Grace recognised the likeness about the two problems. This was very exciting! Most of the above TWO PROBLEMS was published in the December 1979 issue of Mathematics Teaching, a journal of The Association of Teachers of Mathematics, in England.

5. THE NATURAL LOGARITHM IS THE AREA UNDER A CURVE AND AN INFINITE SERIES.

First, what's a logarithm?

You know $10^2 = 100$. The logarithm of 100, base 10 is 2, written $\log_{10} 100 = 2$. The logarithm is the exponent.

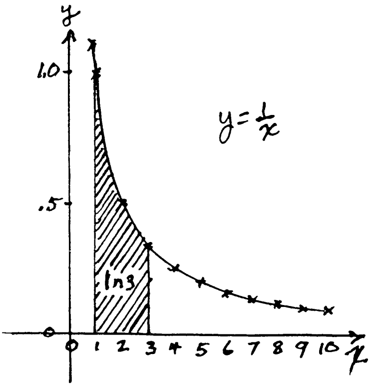
If $2^3 = 8$, then $\log_2 8 = 3$ (the log of 8 base 2 is 3)

The natural log is the log of a number, base e, where e is 2.718 . . . as described in chapter 11.

The logarithms were invented by Napier and independently by Jobst Burgi. They published in 1614 and 1620. Napier was trying to simplify computation. The astronomers Tycho Brahe and Kepler were the first beneficiaries of Napier's work, which changed multiplication problems to addition problems and made their work a lot simpler. Notice that $10^2 \cdot 10^3 = 10^5 = 10^{2+3}$, the exponents add when the powers are multiplied.

On the right is the graph of $y = \frac{1}{x}$ or $x \cdot y = 1$.

FIND THE AREA UNDER THE CURVE FROM $x = 1$ TO $x = 3$. THEN FIND THE AREA FROM $x = 3$ TO $x = 6$. (One of the rules in chapter 6 was 3,120 4,90 6,60 . . . which turns out to be $x \cdot y = 360$. These numbers were obtained by counting the # of images of an object you see between two hinged mirrors vs. the angle between the mirrors. When graphed we get an hyperbola, like $x \cdot y = 1$ at the right).



Using the program on the FX7000G above to count the dots, changed a little, we found the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 3$ which turned out to be 1.0952 (the $\log_e 3 = 1.0986$).

The area from $x = 3$ to $x = 6$ turned out to be .6458 (the $\log_e 2 = .6931$). Adding these, we got the area from 1 to 6 which is 1.7410 (the $\log_e 6 = 1.7917$). The numbers we get by counting dots are close enough.

So $A_{1-3}\left(\frac{1}{x}\right) + A_{3-6}\left(\frac{1}{x}\right) = A_{1-6}\left(\frac{1}{x}\right)$ and this is the same as

$$\log_e 3 + \log_e 2 = \log_e 6 = \log_e (3 \cdot 2).$$

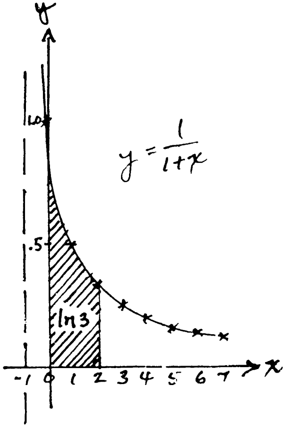
The connection between the addition of areas under the curve and this property of logarithms, that is, the log of the product equals the sum of the logs, was first noticed by A. A. de Sarasa in Gregory's published discovery about the areas in 1647.

Remember Ian's discovery $\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots$

Substitute ^-x for x above or just divide 1 by $1 + x$, and we obtain

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \ldots \text{ for } {}^-1 < x < 1$$

The graph of $y = \frac{1}{1 + x}$ at the right is an hyperbola like $y = \frac{1}{x}$ except it's shifted 1 unit to the left. The area under this curve from $x = 0$ to $x = a$, $A_{0 \text{ to } a}\left(\frac{1}{1 + x}\right) = \log_e (1 + a)$ and if we use the generalizations from the section above, this will equal the integral of the infinite series $1 - a + a^2 - a^3 + \ldots$



Doing each piece separately, we get

$$A_{0 \text{ to } a}\left(\frac{1}{1 + x}\right) = \log_e (1 + a)$$
$$= a - \frac{1}{2} \cdot a^2 + \frac{1}{3} \cdot a^3 - \frac{1}{4} \cdot a^4 + \ldots$$

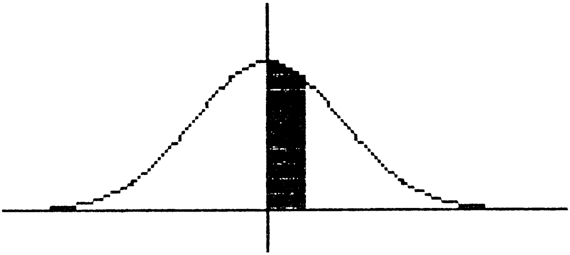
The natural log then, is the area under the curve $y = \frac{1}{x}$ or $y = \frac{1}{1 + x}$ and it's the infinite series shown above (for $-1 < x < 1$)!
CAUTION: You just can't find the $\log_e (1 + 2)$ by putting 2 in for a

in the series above because this series converges only when a is between $^-1$ and 1. Newton found logs of integers in a unique way. First he found the $\log_e 1.1$ by putting .1 in for a in the series. $\log_e (1 + .1) = .1 - \frac{1}{2}(.1)^2 + \frac{1}{3}(.1)^3 - \ldots = .0953$, then similarly got the logs of 1.2, .8 and .9. He then said $2 = \frac{1.2 \cdot 1.2}{.8 \cdot .9}$. Who else would think of writing 2 this way! Newton proceeds to get the log 2 by using the identities for logs: $\log_e 2 = \log_e 1.2 + \log_e 1.2 - (\log_e .8 + \log_e .9)$. This reinforces for me, at least, that we need to write numbers different ways. Ever since I saw Sue Monell do "number names for today's date" with 5- and 6-year-olds 20 years ago, I thought it was a great idea . About a month after Sean made up his rule about the area under the curve x^n, $A_{0 \text{ to } x}(x^n) = \frac{x^{n+1}}{n+1}$, I asked him to graph $y = \frac{1}{x}$ then find the area under it from $x = 1$ to 3 and from $x = 3$ to 6. He started the graph, then thought about his rule. He put $^-1$ in for n and said that his rule doesn't work because he got 0 in the denominator which makes the fraction blow up. Then he proceeded to continue the graph. Tune in for the next exciting episode!

6. IAN'S METHOD OF FINDING THE AREA UNDER THE NORMAL DISTRIBUTION CURVE

During the summer of 1987 I had a student who was getting ready for a course in statistics. I decided to write a program on my PC to plot points (as I had done on the FX7000G). This time I plotted points under the normal distribution curve whose equation is

$$y = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}}$$ and its graph looks like this:

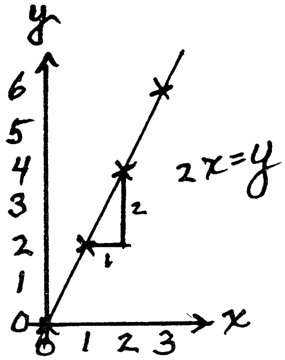


The area under this curve between x = a and x = b is the probability that x lies between a and b. In the use of statistics, the x is transformed into z, which is related to both the mean of the data and the standard deviation.

While writing the program I had a little difficulty and asked Ian to help me. One of the things he did was to change from plotting points to drawing lines, measuring these then adding them up to obtain the area. By doing this, I think Ian invented a method very much the same as Cavalieri's method of indivisibles as described in his book "Geometria indivisibilibus" of 1635 (see Edwards, p.104).

CHAPTER 14: Slopes and The Derivative

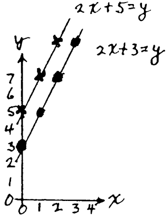
We started graphing in chapter 6; this is an important idea because the derivative can best be shown this way. Let's look at the graph of $2 \cdot x = y$. Notice that if you go one unit to the right, you go up 2 units. This ratio of $\frac{2}{1}$ $\frac{\text{rise}}{\text{run}}$ is called the slope of the graph. The idea of slope appears in many contexts; for example, the slope of a mountain, the pitch of a roof and the gradient of a road. If you were measuring the time an ant moved along the ground, you might let the x-axis represent the time and the y-axis represent the distance travelled by the ant. Then the slope of the graph, 2, would



represent the average rate of speed (distance/time) of the ant, say 2 centimeters per second. The idea of rate or ratio is used in many ways; for example— interest rates (# of dollars per hundred dollars, the word "per" means "divided by"), exchange rate, inflation rate, discount rate, pressure in tires (lbs. per sq. in.), typing speed, film speed, shutter speed, wind speed, population density (# of people per sq. km.), density of materials (mass per unit volume, gm./cc), cost of food in price per oz., frequency of electromagnetic radiation (hertz, or cycles per sec.), speed of light, speed of sound, pollution rate (parts per million), postal rates, shipping rates, rpm (revolutions per minute of a car engine or turntable), pulse rate (beats per minute), crop yield (bu. per hectare), flow of a stream or blood, signal-to-noise ratio, gear ratios on a bike or car, trig ratios (sine, cosine and tangent), probability of an event, batting averages, e.r.a. and growth rates, just to mention a few.

The other thing we can say about the graph and data of the ant is that the distance the ant travels is a function of the time. Of course we won't get a nice simple graph from real data. We talked about functions in chapter 6 also the guess my rules. We should also look at the differences in the numbers. The y-number goes up by 2 when the x-number goes up by 1 which shows up as the slope on the graph! Find the slope of each graph below and look for other things about the graphs:

x	y	
0	0	
		>2
1	2	
		>2
2	4	
		>2
3	6	
		>2
4	8	



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x	y	
0	5	>2
1	7	>2
2	9	>2
3	11	

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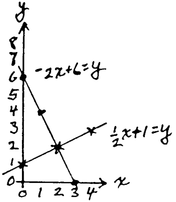
x	y	
0	3	>2
1	5	>2
2	7	>2
3	9	



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x	y	
0	6	>-2
1	4	>-2
2	2	>-2
3	0	

x

x	y	
0	1	> $\frac{1}{2}$
1	$1\frac{1}{2}$	> $\frac{1}{2}$
2	2	> $\frac{1}{2}$
3	$2\frac{1}{2}$	> $\frac{1}{2}$
4	3	> $\frac{1}{2}$

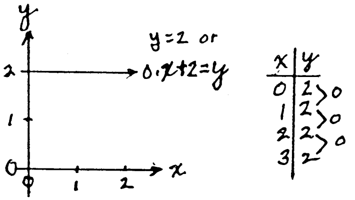
Make up some equations like those above in the form $Ax + B = y$ and then graph them. Get their slopes and look for where the numbers A and B in the equation, show up on the graph.

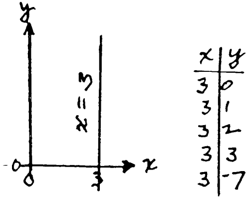
Notice a couple of things about the graphs above. The graphs of $2x + 3 = y$ and $2x + 5 = y$ are parallel lines and both have the same slope, 2. Where do the 3 and 5 in the equations show up on the graph? Could you make up other equations whose graphs are parallel?

The graphs of $^{-}2x + 6 = y$ and $\frac{1}{2}x + 1 = y$ are perpendicular lines and their slopes are negative reciprocals of each other, or $^{-}2 \cdot \frac{1}{2} = {}^{-}1$, the product of their slopes is $^{-}1$. With the negative slope you go down 2 then 1 to the right to get a slope of $\frac{-2}{+1} = {}^{-}2$ or you can go up 2 then 1 to the left to get the same slope of $\frac{+2}{-1} = {}^{-}2$, either way. The fractional slope you get by going up 1, right 2 or for a slope of $\frac{1}{2}$ or you can go up $\frac{1}{2}$, right 1 to get a slope of

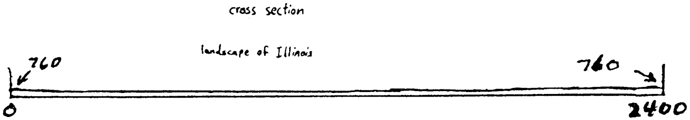
$$\frac{\frac{1}{2}}{1} = \frac{1}{2}.$$

What are the slopes of the two graphs below?

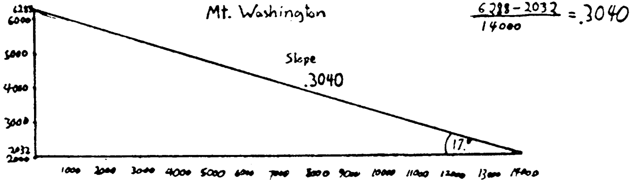




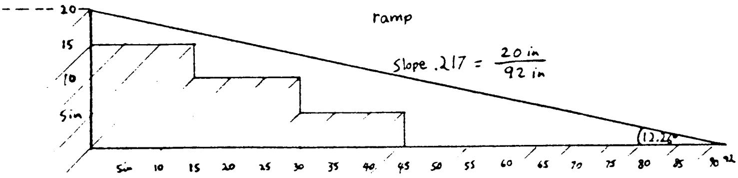
Alec, 13 years old at the time, made a scale drawing of the Illinois landscape from a topographic map of Champaign-Urbana.



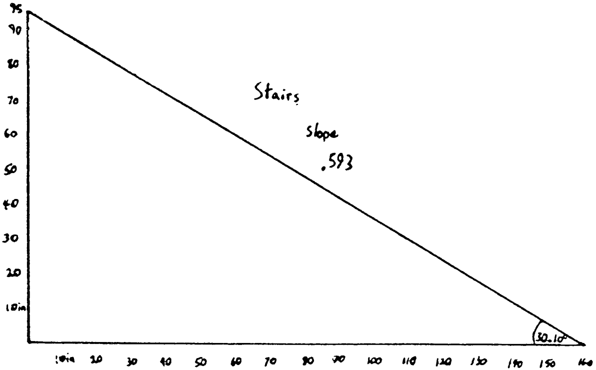
Using a topographic map of New Hampshire, he also found the slope of Mt. Washington to be .304 .Using his dad's calculator he found the angle of elevation to be about 17°.



Alec also found the slope of a wheelchair access ramp and



nearby stairs.



The following are problems from "The Language of Functions and Graphs" published by The Shell Centre for Mathematical Education, University of Nottingham. They lead into the following problem of the derivative very nicely.

Choose the best graph to describe each of the situations listed below. Copy the graph and label the axes clearly with the variables shown in brackets. If you cannot find the graph you want, then draw your own version and explain it fully.

1) The weightlifter held the bar over his head for a few unsteady seconds, and then with a violent crash he dropped it. (height of bar/time)

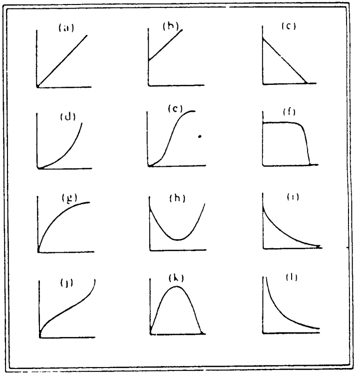
2) When I started to learn the guitar, I initially made very rapid progress. But I have found that the better you get, the more difficult it is to improve still further. (proficiency/amount of practice)

3) If schoolwork is too easy, you don't learn anything from doing it. On the other hand, if it is so difficult that you cannot understand it, again you don't learn. That is why it is so important to pitch work at the right level of difficulty. (educational value/difficulty of work)

4) When jogging, I try to start off slowly, build up to a comfortable speed and then slow down gradually as I near the end of a session. (distance/time)

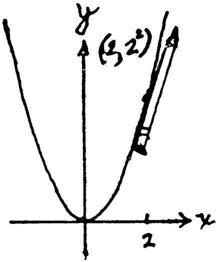
5) "In general, larger animals live longer than smaller animals and their hearts beat slower. With twenty-five million heartbeats per life as a rule of thumb, we find that the rat lives for only three years, the rabbit seven and the elephant and whale even longer. As respiration is coupled with heartbeat—usually one breath is taken every four heartbeats—the rate of breathing also decreases with increasing size. (heart rate/life span)

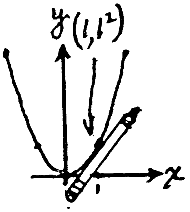
6) As for 5, except the variables are (heart rate/breathing rate)

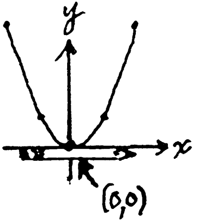


Now make up three stories of your own to accompany three of the remaining graphs. Pass your stories to your neighbour. Can they choose the correct graphs to go with the stories?

Now comes the big change. What's the slope of the curve $y = x^2$? If you take a pencil and put it next to the curve (tangent to the curve), you will see that the slope of the curve changes and gets bigger going to the right. With the straight lines the slope was always the same. So that is our job— find the slope of the curve at any point.

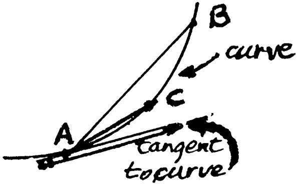






From Jan. 1982 to Feb. 1983 Ian went through a very prodigious period of mathematical discovery (not to diminish his earlier work). In January of 1982, at age 11, upon my suggestion, he started reading W. W. Sawyer's "What is Calculus About?". During this year he figured out the derivative of x^n and discovered the fundamental theorem of the calculus. He did some work on Maclaurin's Theorem, as well as all his work on the binomial theorem (see chapter 9). My role as the teacher was one of giving a few suggestions. I always felt I was not really teaching Ian much at all, but providing an atmosphere in which he could "learn to learn" as he put it. He was always raising questions, pondering the mathematics. I dare say, through about age 14, Ian spent more time thinking about mathematics than 90% of us do in our lifetime, combined! Although Ian is an unusual young man, I have worked with a number of others who are capable of much more than I ever thought possible. If we believe young people can do great things, they will do it—I'm convinced of that.

At some point Ian and I went through the following discussion to get the derivative of x^2, which is the slope of the tangent to the curve $y = x^2$. It's difficult to draw the tangent lines—we certainly can approximate them by putting a pencil next to the curve as we did above. If we try to find the slope of the curve at point A, we connect A and B and find the slope of this straight line as an approximation for the slope at A (obviously too big). By locating point C on the curve between A and B and drawing another straight line, we can get a new slope closer to our final goal. This pattern will be repeated as shown below.

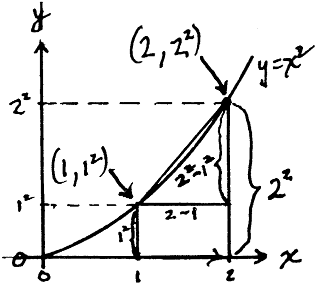


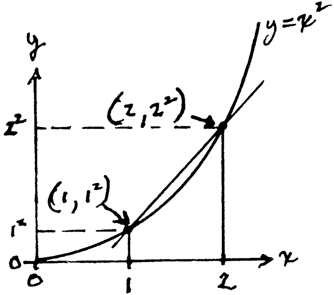
The scales on these graphs will be distorted in order to show the straight lines clearly.

Once we do one of these infinite sequences of slopes, the rest of them will be easy, because there will be a pattern, as usual.

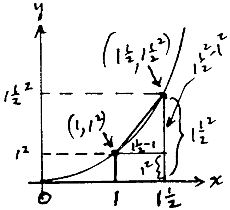
The first problem will be to find the slope of the tangent to the curve $y = x^2$ at the point (1,1). The slope of the straight line from the point (1,1) to $(2,2^2)$ or (2,4) is

$$\text{slope}_{1\text{ to }2} = \frac{\text{rise}}{\text{run}} = \frac{2^2 - 1^2}{2 - 1} = 3$$





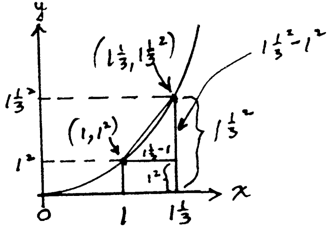
The slope of the straight line from the point (1,1) to $\left(1\frac{1}{2}, \left(1\frac{1}{2}\right)^2\right)$ is



$$\underset{1 \text{ to } 1\frac{1}{2}}{\text{slope}} = \frac{\text{rise}}{\text{run}} = \frac{\left(1+\frac{1}{2}\right)^2 - 1^2}{\left(1+\frac{1}{2}\right) - 1} =$$

$$= \frac{1\frac{1}{4}}{\frac{1}{2}} = 2\frac{1}{2}$$

The slope of the straight line from the point (1,1) to $\left(1\frac{1}{3}, \left(1\frac{1}{3}\right)^2\right)$ is



$$\underset{1 \text{ to } 1\frac{1}{3}}{\text{slope}} = \frac{\text{rise}}{\text{run}} = \frac{\left(1+\frac{1}{3}\right)^2 - 1^2}{\left(1+\frac{1}{3}\right) - 1} =$$

$$= 2\frac{1}{3}$$

Predict the slope of the line from (1,1) to $\left(1\frac{1}{4}, \left(1\frac{1}{4}\right)^2\right)$

Yes, $2\frac{1}{4}$ is the slope from (1,1) to $\left(1\frac{1}{4}, \left(1\frac{1}{4}\right)^2\right)$. You should check it to make sure. Let h be how much we add to the x- coordinate of our first point; it decreases all the time. We can extend the pattern to form the infinite sequence of slopes of straight lines:

$$3,\ 2\frac{1}{2},\ 2\frac{1}{3},\ 2\frac{1}{4},\ 2\frac{1}{5},\ 2\frac{1}{6}, \ldots, 2 + h, \ldots$$

What happening? What is this sequence approaching? 2. Yes.

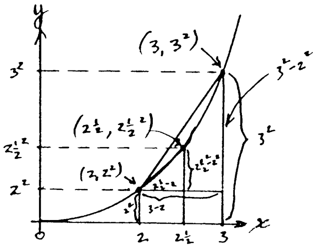
From chapter 9, in the identity $(A + B)^2 = A^2 + 2AB + B^2$ if we put 1 in for A and h in for B we get $(1 + h)^2 = 1 + 2h + h^2$ for the work below.

As $h \to 0$, we say the slope of the tangent is the limit of the sequence of straight lines $= \lim \dfrac{(1+h)^2 - 1^2}{1+h-1} =$

$$\lim \frac{1 + 2h + h^2 - 1}{1 + h - 1} = \lim \frac{2h + h^2}{h} = \lim (2 + h) = 2$$

So the slope of the tangent to the curve $y = x^2$ at the point (1,1) seems to be 2.

Can you find the slope of the tangent to the curve $y = x^2$ at the point $(2,2^2)$. To get started, the slope of the line from $(2,2^2)$ to $(3,3^2)$ is $\dfrac{3^2 - 2^2}{3 - 2} = 5$, then from $(2,2^2)$ to



$$\left(2+\frac{1}{2}, \left(2+\frac{1}{2}\right)^2\right) = \frac{\left(2+\frac{1}{2}\right)^2 - 2^2}{2+\frac{1}{2}-2} = \frac{2\frac{1}{4}}{\frac{1}{2}} = 4\frac{1}{2}$$

You might try $h = .0001$ on a calculator. The infinite sequence of slopes of straight lines from (2,2) looks like this:

$5, 4\frac{1}{2}, 4\frac{1}{3}, 4\frac{1}{4}, \ldots, 4 + h, \ldots$ the limit of which is 4.

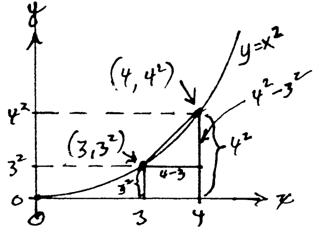
or as $h \to 0$, limit $\frac{(2+h)^2 - 2^2}{2+h-2} = \lim \frac{4+4h+h^2-4}{h} = \lim \frac{4h+h^2}{h} =$ $\lim\,(4+h) = 4$. The slope of the tangent to the curve $y = x^2$ at the point $(2,2^2)$ tentatively is 4.

We'll make a table to show the x-coordinate of the point on the curve and the slope of the tangent at that point:
Can you predict the slope of the tangent to the curve $y = x^2$ at the point $(3,3^2)$?

x	slope of tangent
1	2
2	4
3	?

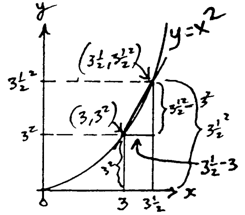
Guess the rule? Let's do this one. The slope of the straight line from the point $(3,3^2)$ to $(4,4^2)$ is

$$\underset{3\text{ to }4}{\text{slope}} = \frac{4^2 - 3^2}{4 - 3} = 7$$



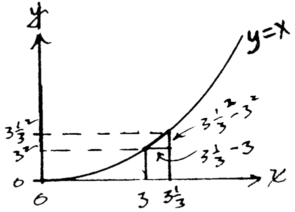
The slope of the straight line from the point $(3,3^2)$ to $\left(3 + \frac{1}{2}, \left(3 + \frac{1}{2}\right)^2\right)$ is

$$\underset{3\text{ to }3\frac{1}{2}}{\text{slope}} = \frac{\left(3 + \frac{1}{2}\right)^2 - 3^2}{\left(3 + \frac{1}{2}\right) - 3} = 6\frac{1}{2}$$



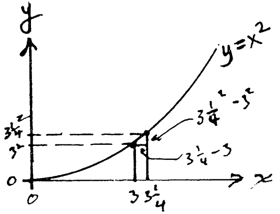
The slope of the straight line from the point $(3,3^2)$ to $\left(3 + \frac{1}{3}, \left(3 + \frac{1}{3}\right)^2\right)$ is

$$\underset{3 \text{ to } 3\frac{1}{3}}{\text{slope}} = \frac{\left(3 + \frac{1}{3}\right)^2 - 3^2}{\left(3 + \frac{1}{3}\right) - 3} = 6\frac{1}{3}$$



The slope of the straight line from the point $(3,3^2)$ to $\left(3 + \frac{1}{4}, \left(3 + \frac{1}{4}\right)^2\right)$ is

$$\underset{3 \text{ to } 3\frac{1}{4}}{\text{slope}} = \frac{\left(3 + \frac{1}{4}\right)^2 - 3^2}{\left(3 + \frac{1}{4}\right) - 3} = 6\frac{1}{4}$$



Try $h = .0001$ instead of $\frac{1}{4}$ on a calculator. The infinite sequence of slopes of straight lines from (3,3) looks like this: 7, $6\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{1}{4}$,. ., $6 + h$, . . . the limit of which is 6.

or as $h \to 0$, limit $\dfrac{(3 + h)^2 - 2^2}{3 + h - 3} = \lim \dfrac{9 + 6h + h^2 - 9}{h} =$

$\lim \dfrac{6h + h^2}{h} = \lim (6 + h) = 6$. The slope of the tangent to the curve $y = x^2$ at the point $(3,3^2)$ is 6.

The table to show the x-coordinate of the point on the curve and the slope of the tangent at that point now looks like this:

x	slope of tangent
1	2
2	4
3	6

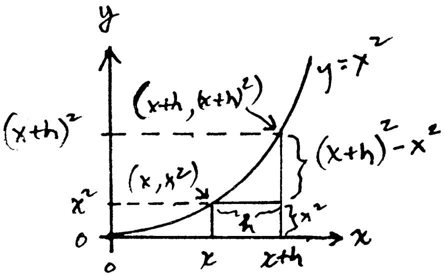
Can you guess the rule? Easy. The slope of the tangent to the curve $y = x^2$ at any point (x,x^2) is $2 \cdot x$.

The slope of the tangent to the curve $y = x^2$ at any point (x,x^2), in other words,

$$\text{the derivative of } x^2 = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{x + h - x} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h} =$$

$$= \lim_{h \to 0} (2x + h) = 2x$$



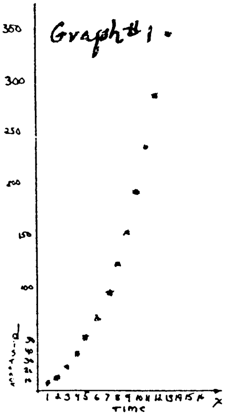
Problem: to see what happens when an object is dropped and to find its speed after 4 ticks of time. A weight attached to paper tape was dropped to the ground. Marks were made on the tape by a recording timer at equal time intervals, like this:

Tape

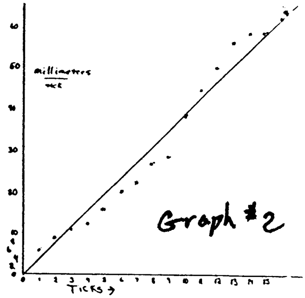
As the weight fell the time between the ticks stayed the same, the marks spread further apart.

After tick #	1	2	3	4	5	6	7	8	9	10	11	12	13
The weight fell (mm)	6	14.5	25	37	53	72.5	94.5	121.5	153	191	235	285	342

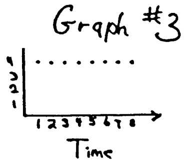
The graph #1 at the right — the distance (in mm.) from the starting point vs. time (in ticks) turns out to look very much like a parabola. The distance travelled increases in each time unit as the weight falls. Galileo first recorded data like this and found that the distance is proportional to the time squared; in other words, in 1, 2, 3, etc. seconds, the distance it travels will be some multiple of 1^2, 2^2, 3^2, . . . Galileo used



his pulse to measure the time in his experiments! The rule that fits this data is a parabola, like $s = \frac{1}{2}at^2$, Where t is the time in sec. s the distance in cm, a would be the acceleration due to gravity, 980 cm/sec^2. The graph #2 at the right shows the change in distance with the change in time. It was obtained by measuring the distance between the marks on the tape. This graph is very close to a straight line such as 2x, which we found for the derivative of x^2. Its slope is about 3.9 mm/tick, its equation, average v = 3.9t



The graph #3 at the right is the change in speed vs. time or the acceleration, whose equation is about av. a = 3.9 $mm/tick^2$. We found the speed after



4 ticks 3 ways:
Method 1. From graph #1: The distance covered between ticks 3 and 4 was 12 mm. The distance between ticks 4 and 5 was 16 mm., so the speed after 4 ticks was between 12 and 16 mm/tick.

Method 2. We went to the graph of change in distance vs. change in time (graph #2). The graph shows the speed after 4 ticks was about 15.6 mm/tick.

Method 3. According to graph #3, the object accelerates about 3.9 mm/tick each tick. Therefore, after 4 ticks, its speed is $4 \cdot 3.9 = 15.6$ mm/tick.

Graphing real data like this never gives perfect graphs like $y = x^2$, $y = 2x$ and $y = 2$. The thing about the mathematics vs. say, physics, is that the mathematics, although theoretical, describes physical reality close enough to enable us to predict very precisely, many things. That's why mathematics is so important to understand, both as a discipline unto itself, as well as for its applications in so many other fields.

Where do we stand on the derivatives, the slopes of the curves so far?
slope of $y = 2$ is 0 (a horizontal line has a slope $= 0$)
slope of $y = 3x$ is 3; slope of $y = ax$ is a
slope of the tangent to the curve $y = x^2$ is $2x$
Try to predict the slopes of the tangent to the curves $y = 5x^2$;
$y = x^2 + 3x + 7$ and $y = x^3$.

The sequence of slopes of straight lines on the curve $y = x^3$ from $(5,5^3)$ to $(5 + h,(5 + h)^3)$ for the curve $y = x^3$ will be

$$\text{for } h = 1, \text{ slope } = \frac{(5 + 1)^3 - 5^3}{5 + 1 - 5} = \frac{6^3 - 5^3}{1} = \frac{91}{1} = 91$$

$$\text{for } h = \frac{1}{2}, \text{ slope } = \frac{\left(5 + \frac{1}{2}\right)^3 - 5^3}{\frac{1}{2}} = \frac{166\frac{3}{8} - 125}{\frac{1}{2}} = \frac{41\frac{3}{8}}{\frac{1}{2}} = 82\frac{3}{4}$$

$$\text{for } h = \frac{1}{3}, \text{ slope } = \frac{\left(5 + \frac{1}{3}\right)^3 - 5^3}{\frac{1}{3}} = \frac{151\frac{19}{27} - 125}{\frac{1}{3}} = \frac{26\frac{19}{27}}{\frac{1}{3}} = 80\frac{1}{9}$$

$$\text{for } h = .0001, \text{ slope } = \frac{(5 + .0001)^3 - 5^3}{.0001} = \frac{.00075002}{.0001} = 75.002$$

We get a sequence 91, $82\frac{3}{4}$, $80\frac{1}{9}$, . . . , 75.002 . .

What's happening?

Using the binomial expansion in chapter 9,
putting $5 \rightarrow A$ and $h \rightarrow B$,

$$(5 + h)^3 = 5^3 + 3 \cdot 5^2 h + 3 \cdot 5^1 h^2 + h^3 \text{ and}$$

in general,

$$\underset{5 \text{ to } (5+h)}{\text{slope}} \text{ of } y = x^3 \text{ is } \frac{(5+h)^3 - 5^3}{h} = \frac{5^3 + 3 \cdot 5^2 h + 3 \cdot 5^1 h^2 + h^3 - 5^3}{h} =$$

$$= \frac{3 \cdot 5^2 h + 3 \cdot 5^1 h^2 + h^3}{h} = 3 \cdot 5^2 + 3 \cdot 5^1 \cdot h + h^2$$

as $h \to 0$, the sequence of slopes of the straight lines approaches the slope of the tangent to the curve $y = x^3$ at the point $(5,5^3)$ and equals $3 \cdot 5^2$.

As $h \to 0$, the sequence of slopes of the straight lines approaches the slope of the tangent to the curve $y = x^3$ at the point $(6,6^3)$ and equals $3 \cdot 6^2$.

Generalizing further,

As $h \rightarrow 0$, the sequence of slopes of the straight lines approaches the slope of the tangent to the curve $y = x^3$ at the point (x,x^3) and equals $3 \cdot x^2$. So the derivative of x^3 is $3x^2$.

So far

function	its derivative
$y = 2$	0
$y = ax$	a
$y = x^2$	$2x$
$y = 5x^2$	$10x$
$y = x^2 + 3x + 7$	$2x + 3$
$y = x^3$	$3x^2$

Ian realized that he could go up and down, starting with $y = x^4$ and used the following notation to show this:

$$y^{-2'} = \frac{x^6}{30}$$

$$y^{-1'} = \frac{x^5}{5}$$

he started here $\quad y^{0'} = x^4$

$$y^{1'} = 4x^3$$

$y^{2'} = 12x^2 \quad$ Ian's notation for the 2nd derivative was $y^{2'}$

$$y^{3'} = 24x^1$$

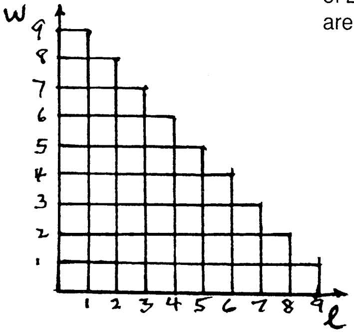
$$y^{4'} = 24$$

and $\quad y^{5'} = 0$

Going down it's nx^{n-1}, going up it's $\frac{x^{n+1}}{n+1}$, the first is the derivative of x^n, the second is the integral (or antiderivative) of x^n. That was exciting!

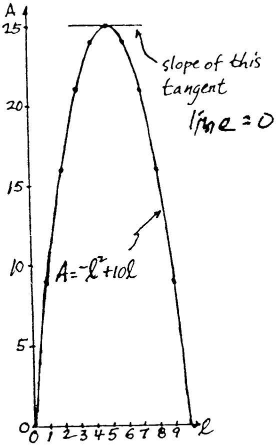
Ian also made up the generalization that the $n + 1$st derivative of t^n is 0; as he showed above, the 5th derivative of x^4 is 0. This is a nice beginning for Taylor's Theorem.

One other application of the derivative which I think is important, besides velocity and acceleration, is using it to find the maximum or minimum of a function. One simple example (which can be done without the calculus as well), is to find the length of the rectangle with the largest area that has a perimeter of, say 20. The diagram at the left shows some of the rectangles which have a perimeter of 20. The lengths, widths and areas are listed below.



ℓ	w	A
1	9	9
2	8	16
3	7	21
4	6	24
5	5	25
6	4	24
7	3	21
8	2	16
9	1	9

The perimeter of a rectangle $= 2 \cdot \ell + 2 \cdot w = 20$, so $w = 10 - \ell$. The Area $= \ell \cdot w$. Then substituting for w, $A = \ell \cdot (10 - \ell) = -\ell^2 + 10 \cdot \ell$. The graph of this equation, a parabola, is at the right. Notice that the area increases, reaches a maximum, then decreases as the length increases. If we take the derivative of the area with respect to the length (find the slope of the tangent to the curve), we get $-2 \cdot \ell + 10$. At the maximum point, the slope of the tangent is 0. So $-2 \cdot \ell + 10 = 0$. Solving this equation for ℓ, we get $\ell = 5$. So the rectangle with the maximum area has a length of 5, which is the square. This idea also applies in 3-D, to surface area and volume.



EPILOGUE

I was thinking of some way to make this book finite, when I listed the infinite sequences and series we've looked at. I was excited. Here they are: some diverging, some converging (if they converge the limit of the sequence or series is given):

Sequences	Lim	Series	Lim
$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$	1	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$	1
$\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \ldots$	$\frac{1}{2}$	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$	$\frac{1}{2}$
$.9, .99, .999, .9999, \ldots$	1	$.9 + .09 + .009 + .0009 \ldots + \ldots$	1
$.18, .1818, .181818, \ldots$	$\frac{18}{99}$	$\frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \ldots$	div
$.037, .03737, .0373737, \ldots$	$\frac{37}{990}$	$.037 + .00037 + .0000037 \ldots$	$\frac{37}{990}$
$1, \frac{12}{9}, \frac{120}{81}, \ldots$	$1\frac{3}{5}$	$1 + \frac{3}{9} + \frac{12}{81} + \frac{48}{729} \ldots$	$1\frac{3}{5}$
$1, \frac{4}{3}, \frac{48}{27}, \frac{192}{81}, \ldots$	div	$\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \ldots$	$\frac{4}{5}$
$1, 4, 9, 16, 25, \ldots$	div	$\left(\frac{4}{3}\right)^0 + \left(\frac{4}{3}\right)^1 + \left(\frac{4}{3}\right)^2 + \ldots$	div
$1, 8, 27, 64, 125, \ldots$	div	$1 + \frac{1}{3} + \frac{12}{27} + \frac{48}{81} + \ldots$	div
$1^2 \cdot 6, 2^2 \cdot 6, 3^2 \cdot 6, 4^2 \cdot 6, \ldots$	div		
$6, 10, 14, 18, \ldots$	div		

Sequences	Lim
6, 5, 4.67, 4.5, 4.4, . . .	4
1, 1, 2, 3, 5, 8, 13, . . .	div
1, 2, 1.5, 1.6, . . ., 1.6 . . .	ϕ
180, 120, 144, 135, 138.46 . . .	137.5
$1, \phi, \phi + 1, 2\phi + 1, 3\phi + 2, \ldots$	div
-1, 11, 4.455, 3.633, 3.358 . . .	3
17, 9, 6.33, 5.44, 5.18 . . .	5
2.6, 3, 3.1, . . .	π
2, 1.5, 1.42, 1.415, . . .	$\sqrt{2}$
1, 1.4, 1.41, 1.414, . . .	$\sqrt{2}$
5, 6.5, 6.3269, 6.3245 . . .	$\sqrt{40}$
1.07, 1.071225, 1.07229 . . .	$e^{.07}$
$\frac{1}{3} \cdot 1^3, \frac{1}{3} \cdot 2^3, \frac{1}{3} \cdot 3^3 \ldots$	div
10, 100, 1000, 10000, . . .	div
5, 7, 9, 11, 13, 15, . . .	div
4, 2, 0, -2, . . .	div
$1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, \ldots$	div

Series	Lim
$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots$	div
$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$	div
$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$	$\frac{1}{9}$
$\frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$	$\frac{6}{7}$
$.43 + .0043 + .000043 + \ldots$	$\frac{43}{99}$
$1 + x + x^2 + x^3 + x^4 + \ldots$	$\frac{1}{1-x}$
(above converges if $\lvert x \rvert < 1$)	
$\frac{A}{B} + \left(\frac{A}{B}\right)^2 + \left(\frac{A}{B}\right)^3 + \ldots$	$\frac{A}{B-A}$
(above converges if $A < B$)	
$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$	$\frac{\pi}{4}$
$1 - x + x^2 - x^3 + x^4 + \ldots$	$\frac{1}{1+x}$

Series	Lim
$1 + .07 + \frac{(.07)^2}{2!} + \frac{(.07)^3}{3!} + \ldots$	$e^{.07}$
$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	e^x
$1 + ia + \frac{(ia)^2}{2!} + \frac{(ia)^3}{3!} + \ldots$	e^{ia}
$1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \ldots$ (a is in radians above)	$\cos a$
$a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \ldots$ (a is in radians above)	$\sin a$
$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots$	$\frac{1}{3}$

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Materials

Calculators: Casio, TI (graphics, scientific, programmable)
Cuisenaire Rods; ETA Cuisenaire 1.800.445.5985
Geoboards; ETA Cuisenaire 1.800.445.5985
Mathematics Calendar; Math Products Plus; P.O. Box 64, San Carlos, CA 94070
Mirrors (2), plastic, 4"x6"; Bert Harrison, Inc., Waltham, MA, 1-800-628-4724; or Cuisenaire Co. 1-800-237-3142
Nautilus Shell; Almark, 76 Northwest 72nd St., Miami, FL 33150; 1-305-756-0431
Pineapple (a Fibonacci one) - go to your market and count the rows..8, 13, 21! People will look at you a little funny
Sunflower stalk and/or head-- start planting!
Ticker-tape (Acceleration Timer): Central Scientific Co.; Franklin Park, IL; 1-800-262-3626; #72702-21
Topographic maps-- U.S. Geological Survey, some bookstores
Tower Puzzle; ETA Cuisenaire; 1-800-445-5985

Videotapes

"Infinite Series By and For 6 year-olds and up"; Running time 24 min.; produced by Don Cohen

"Iteration to Infinite Sequences with 6 to 11 year-olds"; Running time 38 min.; produced by Don Cohen

Computer Software

Derive®, http://education.ti.com/product/software/derive/features/features.html

Logo

Mathematica:Wolfram Research Inc., Champaign, IL 61820: http://www.wri.com

Microsoft Basic" Microsoft Corp., Redmond, WA 1-800-227-4679

DPGraph: by David Parker; http://www.dpgraph.com/subscribe.html

Websites of note

IES wonderful Java applets at http://www.ies.co.jp/math/java

Don's ideas are used in 3 of these at:

http://.ies.co.jp/math/java/comp/itoi/itoi.html (from CH.11-a WOW!) and http://www.ies.co.jp/math/java/trig/sixtrigfn/sixtrigfn.html, all 6 trig functions at once, and http://www.ies.co.jp/math/java/misc/magbox/magbox1.html , Maggie's difference of 2 cubes.

The Math Forum, the best MathED site on the www! http://mathforum.org/ (search for Mathman).

Ron Knott's fine work on Fibonacci numbers and the golden mean at http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html

Xah Lee's great work on curves-especially the equiangular spiral! At http://xanlee.org/SpecialPlaneCurves dir/EquiangularSpiral dir/equiangularSpiral.html

About the author:
Don Cohen was born in Jersey City, N.J. He has taught all ages of students for 37 years, the last 17 of these as co-founder and teacher of The Math Program, with his partner Jerry Glynn.

After 7 years of teaching in a junior high school, he realized there must be a more enjoyable and effective way to teach math. He searched for alternatives. This lead to designing new curriculum for NY State; learning about mathematics and creativity from Bob Davis with The Madison Project; learning what real teaching is about by observing great teachers such as Sue Monell; teaching teachers; working on Plato (a computer-based education system started at the U of IL); all before Don and Jerry invented The Math Program.

One of the most exciting and satisfying events in Don's life was the completion of this book

Calculus By and For Young People
(ages 7, yes 7 and up)

which was reviewed in the Dec. 1988 issue of Scientific American, as well as in other places. He then produced two videotapes to go with the book:

Infinite Series By and For 6 year-olds and up
and
Iteration to Infinite Sequences with 6 to 11 year-olds

$1\frac{1}{2}$ years later Don finished

Calculus By and For Young People -- Worksheets

Don then wrote and published his book **"Changing Shapes With Matrices"** in 1995, which was co-authored and published in Japan in 2001.

Don then created his **A Map to Calculus** which is a 15x18" poster-flowchart overview of how one important mathematical idea is related to the next and to each other, with arrows and boxes. The chapters in his books and videotapes are annotated on the map.

Don has his book **Calculus By and For Young People-Worksheets** put on a CD-ROM for sale in 2006.

He now has **Math by Email** with students around the country. **These are the good old days!**

Don has been blessed with a wonderful wife, three fine sons, and 6 terrific grandchildren and three great-grandchildren. He is a watercolor artist (see cover) and as a friend said, "he takes time to smell the flowers and gives them to people".

"We have not succeeded in answering all your problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things".—from an unknown source, but certainly a kindred spirit. …January 2006

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(the *Scientific American* review continued from the inside front cover)...
The crossings between recreational mathematics, modern calculators and the track of such pioneers as Newton and Euler make this breezy and personal account, more notebook than book, good fun for the mathematically inclined young person and helpful for any adults who seek freer and solid arithmetic teaching".

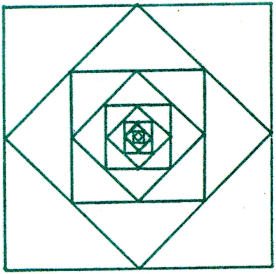
"...how aesthetically pleasing it all is. The beauty of the obvious, once you have been shown!" -- **Dr. Morris Greenberg, England.**

"..very well done." -- **Martin Gardner**.

"My **seven** year old and I have found your book fascinating. " -- Mrs. Kathleen Insler, CT . Jonathan came to Champaign, IL to study with Don for a week in the summers of '89 and '90 and plans to return in the summer of '91. They were the inspiration for Don's **Math by Mail.**

"..I have reviewed the tapes with my eight year old daughter and I must say that I was very impressed with the ease and enthusiasm with which she proceeded to add fractions and predict subsequent terms of the series...". -- Luciano Corazza, Dir. of Academic Programs, The Johns Hopkins U., **Center for the Advancement of Academically Talented Youth (CTY)**

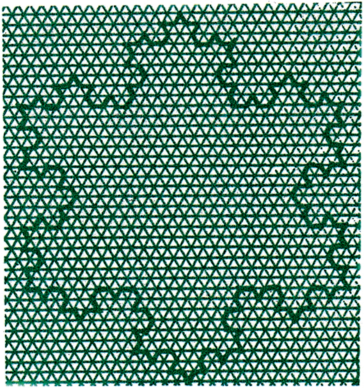
" I've ventured through Mr. Cohen's book, watched his video and experienced the hinking process of his book through his workshop. I only wish Mr. Cohen had been my math teacher, I think I could have been a mathematician! It's an exciting, exploring manner for students to experience; it develops a thinking process that has many applications to life." -- Carol Shaffer, **Specialist in Gifted Education**



Drawing by Justin, 4th grader
Can you find this infinite series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots ?$$

The Snowflake Curve—
how does its perimeter
and area change?



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